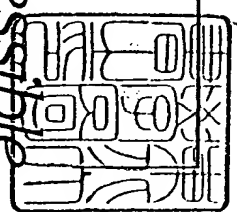


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Compressible

Fluid Flow

Second Edition

Michel A. Saad  
Professor of Mechanical Engineering  
Santa Clara University, California



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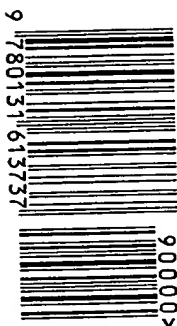
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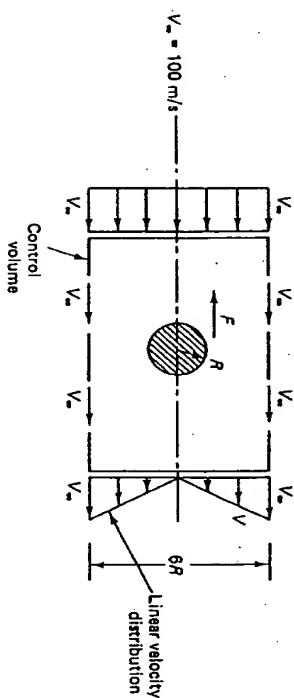


Figure 2.23

the flow field, determine the drag exerted per unit length of cylinder. Assume that  $\rho = 1 \text{ kg/m}^3$ .

- 2.24. An incompressible fluid flows steadily over a wide airfoil as shown in Fig. 2.24. The velocity upstream of the airfoil is uniform at  $V_\infty$  and the velocity profile at the downstream exit of the control volume is symmetric about the  $y = 0$  axis. The  $x$ -velocity component along the upper and lower surfaces of the control surface is constant at  $V_\infty$ . Assuming that the static pressure along the control surface is uniform and neglecting viscous shear effects along the control surfaces, determine the drag force per unit depth acting on the airfoil in terms of  $\rho$ ,  $h$ , and  $V_\infty$ .

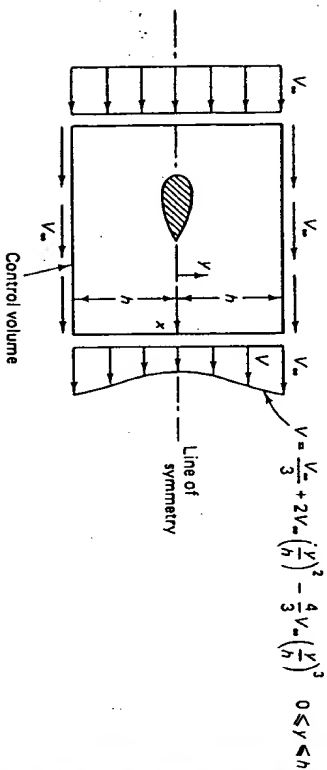


Figure 2.24

- 2.25. For the steady isentropic flow of a compressible fluid, show that
- $$\frac{p}{p_0} = 1 + \frac{1}{\gamma} \ln \left( 1 - \frac{1}{2} M^2 \right)$$

and

$$\frac{\dot{M}}{A} \sqrt{\frac{K_2}{K_1}} = M \sqrt{1 - \frac{1}{2} M^2}$$

where  $K_2 = (1/\rho)(\partial p/\partial p)$  is the isentropic coefficient of compressibility.

## Chapter 3

### Isentropic Flow

#### 3.1 INTRODUCTION

The flow of a compressible fluid in real systems is a complex phenomenon, but it can be interpreted as a combination of several simple types of flow. These "simple" types of flow will be examined as separate entities in order to provide a better insight into the motion of a compressible fluid.

In this chapter the focus of attention will be isentropic flow, where the area of the confining duct changes. In real systems, such as a rocket nozzle, ideal isentropic flow does not occur; simultaneously, there is likely to be some heat interaction with the surroundings and some frictional effects. Adiabatic flow through a variable-area duct approaches isentropic flow if the walls of the duct are smooth and if the fluid has zero viscosity so that no irreversible effects occur. The main changes in flow, then, are caused by variation of the cross-sectional area. When large volumes of gas flow in a constant-area duct and only negligible amounts of heat are transferred, the flow is treated as though it were adiabatic with friction (Chapter 5). Similarly, when a large amount of heat is transferred to a gas flowing in a constant-area duct, and frictional effects are minor, the flow may be considered frictionless and nonadiabatic (Chapter 6). These idealized flow

systems serve as the basis for evaluating and comparing actual flow systems and provide solutions sufficiently accurate for many engineering applications.

The analysis in these first chapters describes one-dimensional<sup>†</sup> steady flow in which the fluid behaves like a perfect gas. The more general case, where there is nonadiabatic, frictional flow in a variable-area duct, is discussed in Chapter 6. Despite their inherent simplicity, one-dimensional steady state models can accurately represent numerous flow systems.

### 3.2 FLOW IN A DUCT OF VARYING CROSS-SECTIONAL AREA

Consider steady-state, one-dimensional flow of a compressible fluid in a duct of varying cross-sectional area, as shown in Fig. 3.1. If the flow is adiabatic and frictionless (i.e., isentropic), the static properties change, owing to variations in the cross-sectional area, but stagnation properties remain unchanged. In a *nozzle*, gradual area changes occur in such a way that the velocity of flow constantly increases. Portions of the nozzle may be convergent or divergent. In a convergent-divergent nozzle, the flow passage decreases to a minimum cross section, known as the throat. Then the flow passage increases in the divergent portion. In a *diffuser*, on the other hand, the flow undergoes deceleration, and this is accompanied by an increase in pressure. When the flow is frictionless, no work is done along the boundary of the passage as the fluid flows through a nozzle or a diffuser.

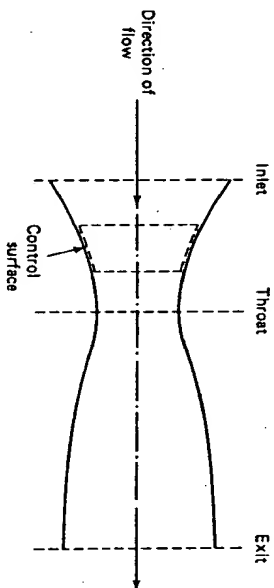


Figure 3.1 Flow in a varying-area duct.

According to continuity relationships, when an incompressible fluid flows in steady state through a duct, the product of the cross-sectional area and the velocity is constant. Consequently, the fluid will not accelerate unless the area decreases. If the area must increase, as in a diffuser, the fluid will then decelerate.

<sup>†</sup> Note that one-dimensional treatment gives no information about the variations of properties normal to the streamlines. Only one coordinate is needed to describe the spatial variation of the dependent variables.

On the other hand, if a compressible fluid flows through a duct, properties of the flow depend on both the contour of the passageway and the Mach number.

Consider as a control volume a thin cylinder, as shown in Fig. 3.1. According to steady-state continuity relationships:

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

If variations in height are neglected, Euler's equation can be written as:

$$\frac{1}{\rho} = -V \frac{dV}{dp}$$

According to this equation, a deceleration results in an increase in pressure; conversely, an acceleration results in a decrease in pressure.

Density, from Euler's equation, is replaced in the continuity equation, giving:

$$-V \frac{dV}{dp} dp + \frac{dA}{A} + \frac{dV}{V} = 0$$

which can be rearranged as:

$$-V^2 \frac{dp}{dp} \frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0$$

But since the flow is isentropic, the term  $V^2(dp/dp)$  is equal to  $M^2$ . Hence, velocity and area are related as follows:

$$\frac{dV}{V} = \frac{1}{M^2 - 1} \frac{dA}{A} \quad (3.1)$$

Alternately, by combining Euler's equation and Eq. (3.1), pressure and area are related as follows:

$$\frac{dp}{\rho V^2} = \frac{1}{1 - M^2} \frac{dA}{A} \quad (3.2)$$

Also, density is related to area or velocity in the following way:

$$\frac{d\rho}{\rho} = \frac{M^2}{1 - M^2} \frac{dA}{A} = -M^2 \frac{dV}{V} \quad (3.3)$$

Equation (3.1) indicates that the relative change in velocity with respect to the relative change in cross-sectional area is:

$$\frac{dV}{V} = \frac{1}{M^2 - 1} \frac{dA}{A} \quad (3.1a)$$

If  $M < 1$ , this ratio is negative and the velocity varies inversely with the cross-sectional area. If  $M > 1$ , this ratio is positive and the velocity varies in the same sense as the cross-sectional area. A plot of Eq. (3.1a) is shown in Fig. 3.2. In Fig. 3.3 the variation of relative density according to Eq. (3.3) with Mach number is shown.

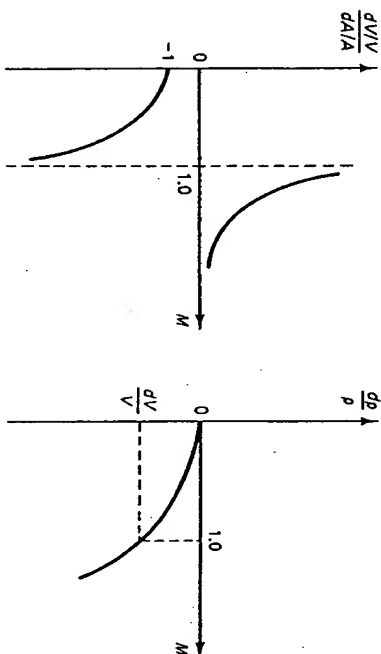


Figure 3.2 Variation of relative velocity with cross-sectional area.

Figure 3.3 Variation of relative density with Mach number ( $dV/V$  positive).

As indicated by Eq. (3.1), whether a fluid accelerates or decelerates at any point in a duct depends not only on the cross-sectional area of the duct at that point but also on whether the Mach number of the stream is more or less than 1. A change of area produces opposite effects on subsonic and supersonic flows. A convergent duct forms a nozzle if  $M < 1$ , but it is a supersonic diffuser if  $M > 1$ . When  $M$  is different from 1 and  $dA = 0$ , then according to Eqs. (3.1), (3.2), and (3.3)  $dV = 0$ ,  $dp = 0$ , and  $d\rho = 0$ . However, no changes of velocity, pressure, or density also occur when  $dA = 0$  and  $M = 1$ , conditions which can exist at the throat of a nozzle. For subsonic flow in a converging duct, the flow cannot accelerate beyond Mach number 1. But if the converging duct is followed by a diverging section, the flow, depending on the exit pressure, may accelerate from a subsonic velocity at the inlet to supersonic velocity at the exit. The Mach number downstream of the throat exceeds 1. At the throat of a nozzle the transition be-

TABLE 3.1

	$M < 1$			$M > 1$		
$dA +$	$dV -$	$dp +$	$dp +$	$dV +$	$dp -$	$dp -$
$dA -$	$dV +$	$dp -$	$dp -$	$dV -$	$dp +$	$dp +$

tween the converging and diverging portions must occur smoothly, so that the flow accelerates continuously. Note that in isentropic flow, sonic velocity does not occur in either a converging or diverging duct; it occurs only at the throat. These results are summarized in Table 3.1 and are also shown in Fig. 3.4.

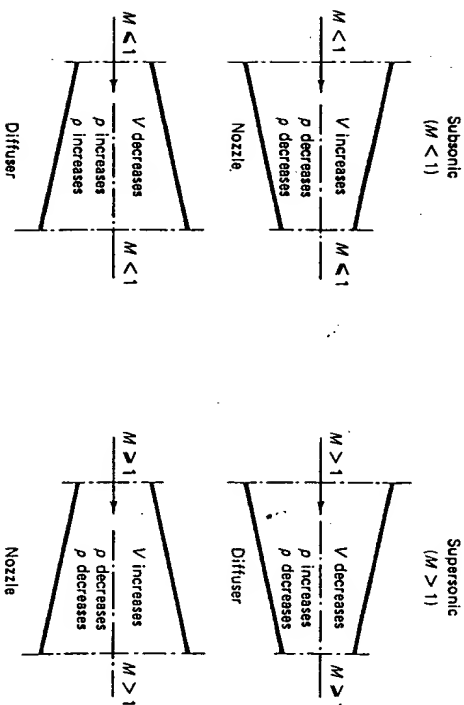


Figure 3.4 Variation of velocity, pressure, and density due to area change for subsonic and supersonic flows.

### 3.3 PROPERTY RELATIONS FOR ISENTROPIC FLOW OF A PERFECT GAS

At this point, flow properties of a perfect gas with constant specific heats will be developed. The relations, expressed in dimensionless form, are based on stagnation properties and Mach number.

From energy relationships, Eq. (1.32), static temperature and stagnation temperature are related as follows:

$$\frac{T_0}{T} = 1 + \frac{\gamma}{2} M^2$$

But:

$$\frac{V^2}{\gamma R T} = M^2 \quad \text{and} \quad c_p = \frac{\gamma R}{\gamma - 1}$$

where  $\gamma$  is the specific heat ratio. Therefore:

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \quad (3.4)$$

This relationship is valid for adiabatic flow and for isentropic flow. Note also that  $T_0$  is the same for all points in the flow, provided that the flow is adiabatic.

When a perfect gas flows isentropically, its pressure and density are related to temperature in the following ways:

$$\frac{p_0}{p} = \left( \frac{T_0}{T} \right)^{\gamma/(\gamma-1)} \quad \text{and} \quad \frac{\rho_0}{\rho} = \left( \frac{T_0}{T} \right)^{1/(\gamma-1)}$$

By combining these with Eq. (3.4), pressure and density can be expressed in terms of Mach number:

$$\frac{p_0}{p} = \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{\gamma/(\gamma-1)} \quad (3.5)$$

and

$$\frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{1/(\gamma-1)} \quad (3.6)$$

$p_0$  and  $\rho_0$  are the same at all points in the flow, provided that the flow is isentropic.

The speed of sound also is a function of temperature:

$$\frac{c_0}{c} = \sqrt{\frac{T_0}{T}}$$

Therefore sonic speed can be expressed as a function of the Mach number:

$$\frac{c_0}{c} = \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{1/2} \quad (3.7)$$

where  $c$  is the speed of sound in the gas, while  $c_0$  is the speed of sound when the gas is at the stagnation temperature. When a flow is decelerated adiabatically to zero velocity, the temperature of the fluid becomes  $T_0$ . However, the fluid will not necessarily exist at the isentropic stagnation state. Unless deceleration is accomplished isentropically, the actual stagnation pressure and stagnation density fall below the isentropic stagnation values.

By means of Eqs. (3.4) through (3.6), values of  $T_0/T$ ,  $p_0/p$ , and  $\rho_0/\rho$  can be tabulated<sup>†</sup> for several values of  $\gamma$  and for any Mach number. Figure 3.5 shows these ratios as a function of  $M$  for  $\gamma = 1.4$ .

Properties of a fluid when the gas is flowing at Mach 1 are called the *critical properties*. They are usually identified by means of an asterisk (\*) to distinguish them from properties at other Mach values. Like stagnation properties, they are used as reference in describing properties at different sections of the flow. Equations describing critical properties referred to stagnation properties are obtained

<sup>†</sup> See, for example, J. H. Keenan and J. Kaye, *Gas Tables* (New York: John Wiley & Sons, Inc., 1945). An abstract of similar tables is given in Table A2 of the Appendix.

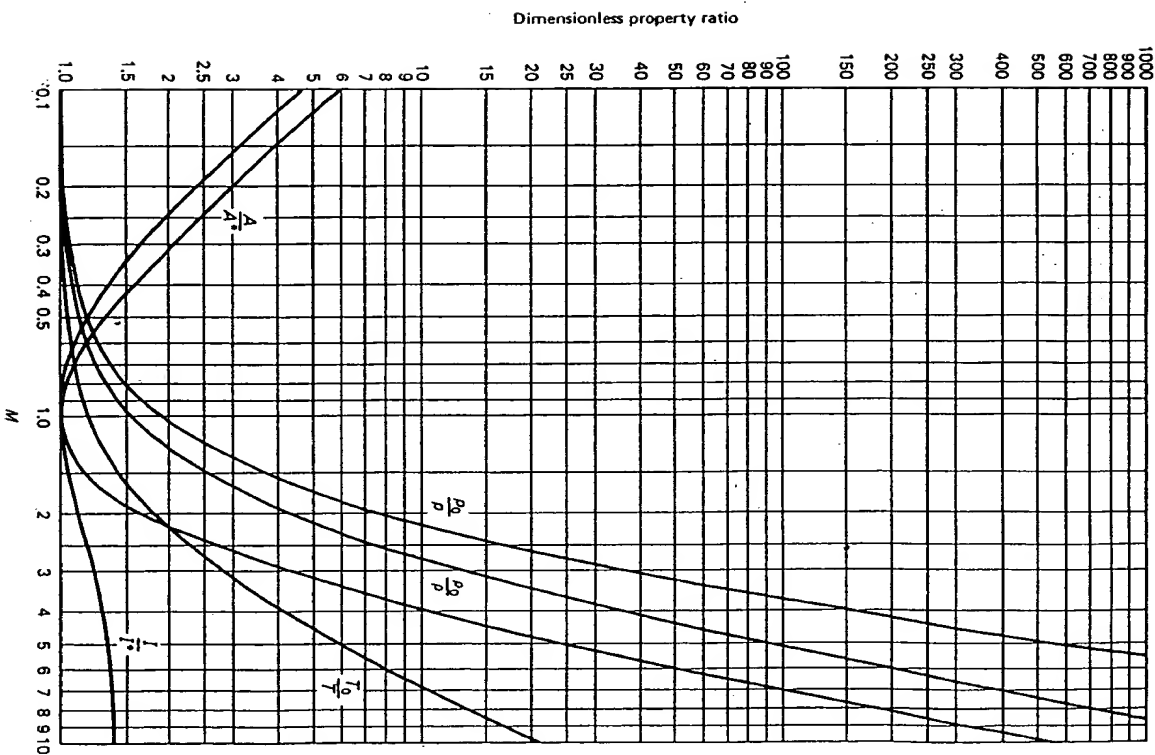


Figure 3.5 Isentropic relations ( $\gamma = 1.4$ ).

from Eqs. (3.4) to (3.7) by substituting  $M = 1$ :

$$\frac{T^*}{T_0} = \frac{c^{*2}}{c_0^2} = \frac{2}{\gamma + 1} \quad (= 0.8333 \text{ when } \gamma = 1.4) \quad (3.8)$$

$$\frac{p^*}{p_0} = \left( \frac{2}{\gamma + 1} \right)^{\gamma/(\gamma-1)} \quad (= 0.5283 \text{ when } \gamma = 1.4) \quad (3.9)$$

$$\frac{\rho^*}{\rho_0} = \left( \frac{2}{\gamma + 1} \right)^{1/(\gamma-1)} \quad (= 0.6339 \text{ when } \gamma = 1.4) \quad (3.10)$$

$$\frac{c^*}{c_0} = \left( \frac{2}{\gamma + 1} \right)^{1/2} \quad (= 0.912 \text{ when } \gamma = 1.4) \quad (3.11)$$

Properties of the fluid at any point and referred to critical properties are obtained by combining Eqs. (3.4) through (3.7) with Eqs. (3.8) through (3.11):

$$\frac{T}{T^*} = \frac{\gamma + 1}{2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right)} \quad (3.12)$$

$$\frac{p}{p^*} = \left[ \frac{\gamma + 1}{2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right)} \right]^{\gamma/(\gamma-1)} \quad (3.13)$$

$$\frac{\rho}{\rho^*} = \left[ \frac{\gamma + 1}{2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right)} \right]^{1/(\gamma-1)} \quad (3.14)$$

$$\frac{c}{c^*} = \left[ \frac{\gamma + 1}{2 \left( 1 + \frac{\gamma - 1}{2} M^2 \right)} \right]^{1/2} \quad (3.15)$$

From energy considerations, velocity is expressed by:

$$V = \sqrt{2c_p(T_0 - T)} = \sqrt{\frac{2\gamma}{\gamma - 1} R(T_0 - T)} \quad (3.16)$$

The maximum velocity attainable is reached when the absolute temperature of the fluid is zero:

$$\begin{aligned} V_{\max} &= \sqrt{2h_0} = \sqrt{\frac{2\gamma}{\gamma - 1} RT_0} \\ &= \sqrt{\frac{2\gamma}{\gamma - 1} \frac{p_0}{\rho_0}} = c_0 \sqrt{\frac{2}{\gamma - 1}} \\ &= 2.24c_0 \quad (\gamma = 1.4) \end{aligned} \quad (3.17)$$

Note that  $V_{\max}$  is always finite; at  $V_{\max}$ , however, the Mach number is infinite, because the sonic velocity at that temperature is zero.

The highest velocity attainable with 293 K air is therefore:

$$V_{\max} = \sqrt{\frac{2 \times 1.4 \times 287.04}{0.4}} T_0 = 44.82 \sqrt{T_0} = 767.2 \text{ m/s}$$

The Mach number bears an inverse relationship to the gas temperature, so that a gas at low temperature can have a very large Mach number even though its velocity is not so large. For this reason, velocity is often expressed in terms of a dimensionless number,  $M^*$ , rather than in terms of  $M$ .  $M^*$  is defined as:

$$M^* = \frac{V}{c^*} \quad (3.18)$$

where  $c^*$  is the speed where the Mach number is equal to 1; it is a constant for any given stagnation conditions.  $M^*$  has the advantage of being finite at very large values of  $M$ . While  $M$  relates velocities associated with one particular location,  $M^*$  relates a local velocity with the velocity at a different point in the system. To derive an expression relating  $M$  with  $M^*$ , Eq. (3.11) is combined with Eq. (1.63), giving:

$$\frac{2}{\gamma - 1} \frac{c^2}{c^{*2}} + \frac{V^2}{c^{*2}} = \frac{\gamma + 1}{\gamma - 1} \quad (3.19)$$

But:

$$M^* = \frac{V}{c^*} \quad \text{and} \quad M = \frac{V}{c}$$

so that:

$$\frac{M^*}{M} = \frac{\frac{V}{c^*}}{\frac{V}{c}} = \frac{c}{c^*} \quad (3.20)$$

By substituting Eq. (3.20) into Eq. (3.19),  $M^*$  can be expressed in terms of  $M$  as follows:

$$\begin{aligned} M^* &= \frac{M \sqrt{\frac{\gamma + 1}{2}}}{\sqrt{1 + \frac{\gamma - 1}{2} M^2}} \\ &= 2.24c_0 \quad (\gamma = 1.4) \end{aligned} \quad (3.21)$$

Alternatively,  $M$  may be expressed as a function of  $M^*$ :

$$M = \frac{M^* \sqrt{\frac{2}{\gamma + 1}}}{\sqrt{1 - \frac{\gamma - 1}{\gamma + 1} M^{*2}}} \quad (3.22)$$

When  $M$  is 0,  $M^*$  is also 0; when  $M$  is less than 1,  $M^*$  is also less than 1; when  $M$  is 1,  $M^*$  is 1; and when  $M$  is greater than 1,  $M^*$  is also greater than 1. But

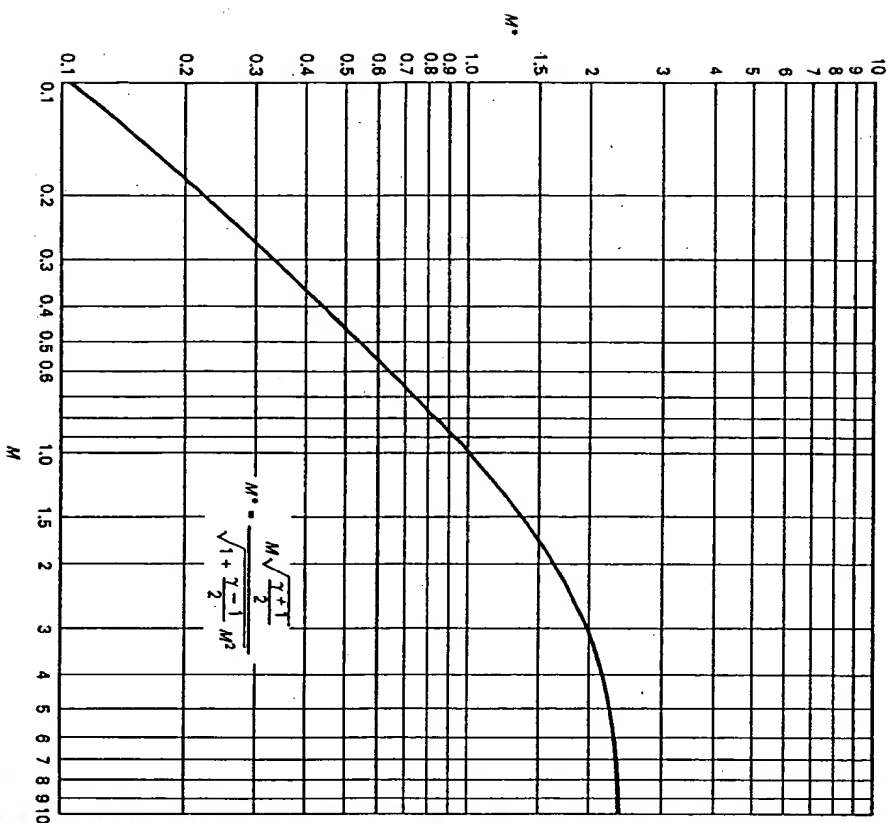


Figure 3.6  $M^*$  versus  $M$  for  $\gamma = 1.4$ .

when  $M$  is infinite,

$$M^* = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \quad (= 2.4495 \text{ for } \gamma = 1.4)$$

A plot of  $M^*$  versus  $M$  is shown in Fig. 3.6.

### 3.4 MASS RATE OF FLOW IN TERMS OF MACH NUMBER

The mass flow rate per unit area (mass flux) for the flow is:

$$G = \frac{\dot{m}}{A} = \rho V$$

But for a perfect gas, since  $\rho = p/RT$  and  $c = \sqrt{\gamma RT}$ , therefore:

$$G = p \left( \frac{V}{c} \right) \sqrt{\frac{\gamma}{RT}}$$

Substituting for  $p$  and  $T$  from Eqs. (3.4) and (3.5) and noting that  $V/c = M$ , the mass flux may be expressed in terms of the Mach number and stagnation properties as:

$$G = \frac{\dot{m}}{A} = \frac{p_0}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}} \frac{M}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{(\gamma + 1)/(2(\gamma - 1))}} \quad (3.23)$$

The rate of mass flow is proportional to the stagnation pressure, but it is inversely proportional to the square root of the stagnation temperature. Equation (3.23) can be written in the form:

$$G = \frac{\dot{m}}{A} = C \frac{p_0}{\sqrt{T_0}}$$

where the coefficient  $C$  is given by:

$$C = \sqrt{\frac{\gamma}{R}} \frac{M}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{(\gamma + 1)/(2(\gamma - 1))}}$$

Values of  $C$  for  $\gamma = 1.4$  and  $R = 287 \text{ J/kg K}$  (air) are given in the following table as a function of  $M$ :

$M$	$C \times 10^3$	$M$	$C \times 10^3$	$M$	$C \times 10^3$
0.1	0.694	1.5	3.436	4.5	0.244
0.2	1.364	2	2.395	5	0.162
0.4	2.542	2.5	1.533	6	0.075
0.6	3.402	3.0	0.954	7	0.039
0.8	3.893	3.5	0.595	8	0.021
1.0	4.042	4.0	0.377	10	0.0075

In Fig. 3.7 the parameter  $\dot{m}\sqrt{T_0}/\rho_0 A$  is plotted as a function of  $M$ , for  $\gamma = 1.4$  and  $R = 287.04 \text{ J/kg K}$ . The maximum value of  $G$  occurs at the section of minimum flow area when  $M = 1$ . This can be ascertained by differentiating Eq. (3.23) with respect to  $M$  and setting the result to zero. If  $M = 1$  at the throat, where the area is minimum, the mass flow rate per unit area is:

$$G^* = \left(\frac{\dot{m}}{A}\right)_{\max} = \frac{\dot{m}}{A^*} = \frac{P_0}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \quad (3.24)$$

Equation (3.24) is valid for both isentropic flow and nonisentropic flow. However, the area  $A^*$  is a constant only in isentropic flow. If  $\gamma = 1.4$  and  $R = 287.04 \text{ J/kg K}$ , then:

$$G^* = \left(\frac{\dot{m}}{A}\right)_{\max} = 0.04042 \frac{P_0}{\sqrt{T_0}} \quad (3.25)$$

where  $\dot{m}$  is in kg/s,  $A$  in  $\text{m}^2$ ,  $P_0$  in Pa, and  $T_0$  in K. The area ratio may be expressed in terms of  $\gamma$  and  $M$  by dividing Eq. (3.24) by Eq. (3.23):

$$\frac{A}{A^*} = \frac{G^*}{G} = \frac{1}{M} \left[ \left( \frac{2}{\gamma+1} \right) \left( 1 + \frac{\gamma-1}{2} M^2 \right) \right]^{(\gamma+1)/(2(\gamma-1))} \quad (3.26)$$

The ratio  $A/A^*$  is the cross-sectional area where the stream is at the Mach number  $M$  divided by the cross-sectional area where  $M = 1$ . The value  $A/A^*$  is never less than unity, and its minimum value of unity occurs at  $M = 1$ . Figure 3.5 shows the ratio  $A/A^*$  as a function of Mach number for  $\gamma = 1.4$ . Each area ratio corresponds to two values of Mach number, one of which applies to subsonic flow and the other to supersonic flow. Note that the area  $A^*$  need not actually exist but can be considered to be the area that would be necessary to accelerate or decelerate the flow isentropically to  $M = 1$ . Values of  $A/A^*$  are listed in gas tables (Table A2 of the Appendix) as a function of Mach number for  $\gamma = 1.4$ .

An expression that relates flow area with Mach number can be obtained

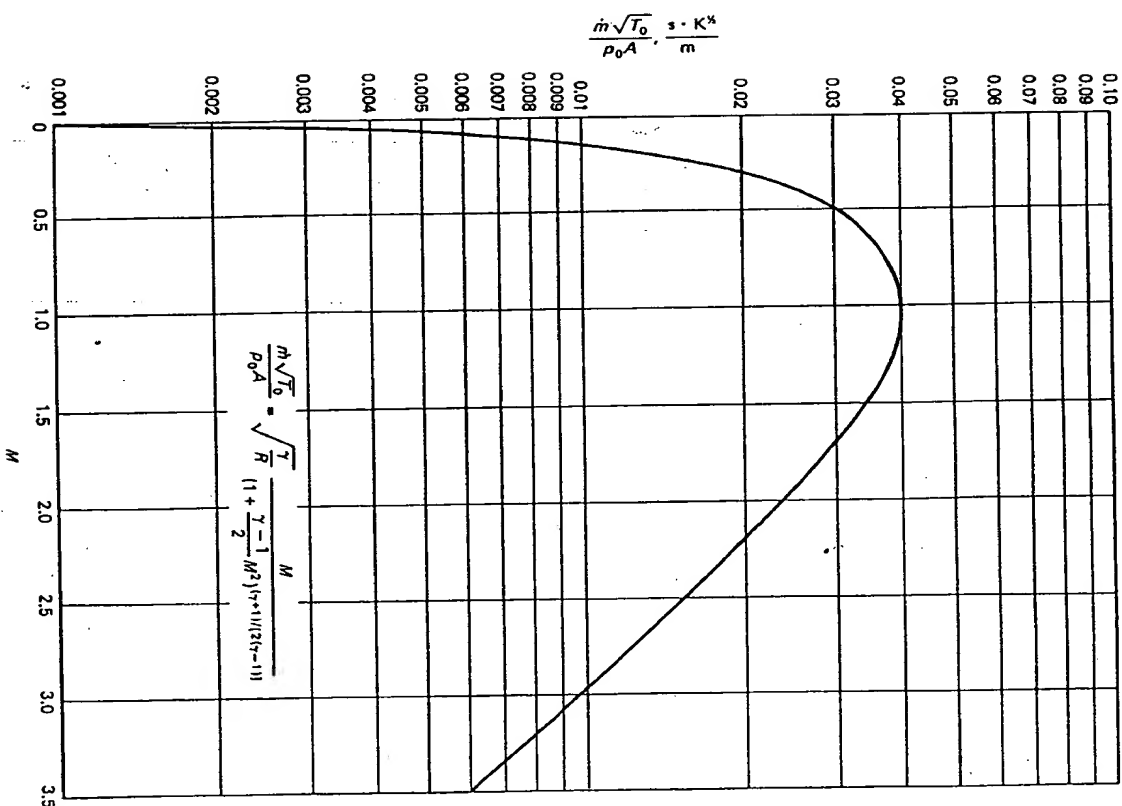


Figure 3.7 The parameter  $\dot{m}\sqrt{T_0}/\rho_0 A$  as a function of  $M$  ( $\gamma = 1.4$ ).

from Eq. (3.1). When the right-hand side of this equation is multiplied and divided by  $c^*$ , the relative area change becomes:

$$\frac{dA}{A} = (M^2 - 1) \frac{d\left(\frac{V}{c^*}\right)}{\frac{V}{c^*}} = (M^2 - 1) \frac{dM^*}{M^*}$$

Substituting for  $M^*$  from Eq. (3.21), the preceding equation becomes:

$$\frac{dA}{A} = \frac{M^2 - 1}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM}{M} \quad (3.27)$$

This equation can also be written as:

$$\frac{dM}{dx} = \frac{M \left(1 + \frac{\gamma - 1}{2} M^2\right)}{A(M^2 - 1)} \frac{dA}{dx} \quad (3.27a)$$

where  $dx$  is the differential of distance along the direction of flow. In Fig. 3.8 the variation of the Mach number and the pressure are shown as a function of  $x$  along a convergent-divergent nozzle. At the throat  $dA/dx = 0$ , and according to Eq. (3.27a)  $dM/dx = 0$  unless  $M = 1$ . Also at  $M = 1$ ,  $dM/dx = \infty$  unless  $dA/dx = 0$ . When both  $M = 1$  and  $dA/dx = 0$ , then Eq. (3.27a), by using l'Hospital's rule, gives:

$$\left(\frac{dM}{dx}\right)^* = \pm \frac{1}{2} \sqrt{-\frac{\gamma + 1}{A^2} \left(\frac{d^2 A}{dx^2}\right)^*} \quad (3.28)$$

According to this equation the slope  $dM/dx$  at the critical point exists if  $(d^2 A/dx^2)^*$  is negative or zero. The value of  $(d^2 A/dx^2)^*$  must, however, be negative for subsonic flow to accelerate continuously to become supersonic. This corresponds to curve  $c$  in Fig. 3.8. By integrating Eq. (3.27) and by applying the condition of  $A = A^*$  at  $M = 1$ , the area ratio becomes:

$$\frac{A}{A^*} = \frac{1}{M} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{(\gamma + 1)/(2(\gamma - 1))}$$

which is the same as Eq. (3.26).

### Example 3.1

Find the stagnation temperature, pressure, and density of air at the nose of an airplane traveling at a speed of 200 m/s. The air temperature is 288 K and the pressure is 1 atm (101.3 kPa). Assume steady isentropic flow.

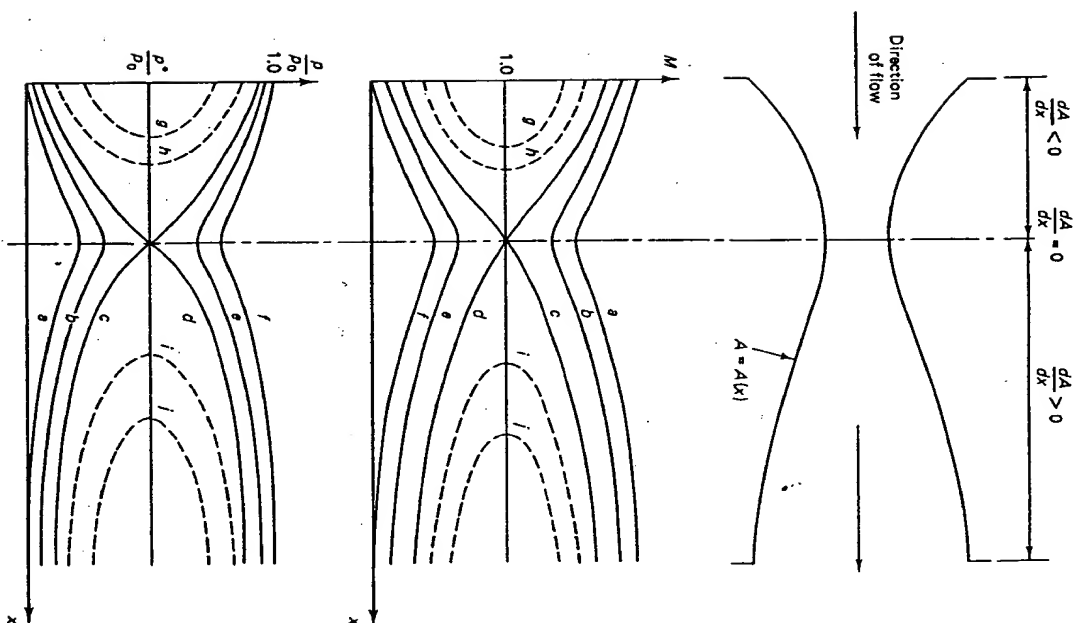


Figure 3.8 Variation of Mach number and pressure as a function of  $x$  in a convergent-divergent duct according to Eq. (3.27a). (Dashed curves have no physical meaning and curve  $d$  is not possible.)

**Solution** The Mach number is:

$$M = \frac{V}{20.1\sqrt{T}} = \frac{200}{20.1\sqrt{288}} = 0.59$$

From Eqs. (3.4) through (3.6), the stagnation temperature, pressure, and density are:

$$T_0 = T \left( 1 + \frac{\gamma-1}{2} M^2 \right) = 288[1 + 0.2 \times (0.59)^2] = 308 \text{ K}$$

$$p_0 = p \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{\gamma/(\gamma-1)} = 101.3[1 + 0.2 \times (0.59)^2]^{1.4/0.4} = 128.21 \text{ kPa}$$

$$\rho_0 = \rho \left( 1 + \frac{\gamma-1}{2} M^2 \right)^{1/(\gamma-1)} = \frac{1.013 \times 10^3}{287 \times 288} [1 + 0.2 \times (0.59)^2]^{1/0.4} = 1.45 \text{ kg/m}^3$$

These results can also be obtained by the use of gas tables.

### Example 3.2

Air flows isentropically at the rate of 1 kg/s through a nozzle. The stagnation temperature is 310 K and the stagnation pressure is 810 kPa. If the exit pressure is 101.3 kPa, determine:

- The throat area.
- The exit Mach number.
- The exit velocity.

**Solution**

(a) The critical pressure according to Eq. (3.9) is:

$$\frac{p^*}{p_0} = 0.528$$

Therefore  $p^* = 427.68 \text{ kPa}$ . Since  $p_{\text{exit}}$  is lower than  $p^*$  and the flow is isentropic, the nozzle must be convergent-divergent. The flow is sonic at the throat and expands to supersonic speeds in the diverging portion of the nozzle.

The critical properties at the throat are:

$$T^* = \left( \frac{2}{\gamma+1} \right) T_0 = 0.833(310) = 258.53 \text{ K}$$

$$\rho^* = \frac{p^*}{RT^*} = \frac{4.277 \times 10^5}{287 \times 258.53} = 5.76 \text{ kg/m}^3$$

$$V^* = c^* = \sqrt{\gamma RT^*} = \sqrt{1.4 \times 287 \times 258.53} = 323.2 \text{ m/s}$$

The throat area according to the continuity equation is:

$$A^* = \frac{\dot{m}}{\rho^* V^*} = \frac{1}{5.76 \times 323.2} = 5.37 \times 10^{-4} \text{ m}^2$$

The area of the throat may also be determined from Eq. (3.25):

$$A^* = \frac{\dot{m} \sqrt{T_0}}{0.04042 p_0} = \frac{1 \sqrt{310}}{0.04042 \times 8.1 \times 10^5} = 5.38 \times 10^{-4} \text{ m}^2$$

(b) The exit Mach number is determined from the pressure ratio:

$$\frac{p_e}{p_0} = \frac{101.3}{810} = 0.125$$

From Table A2,  $M_e = 2.015$ .

(c) At  $M_e = 2.015$ :

$$\frac{T_e}{T_0} = 0.552$$

from which  $T_e = 0.551(310) = 171.12 \text{ K}$ . The exit velocity is:

$$\begin{aligned} V_e &= M_e c_e = M_e \sqrt{\gamma R T_e} \\ &= (2.015)(20.1\sqrt{171.12}) = 529.81 \text{ m/s} \end{aligned}$$

### Example 3.3

Find an expression for the pressure-time history in "blowing down" a pressurized gas tank of volume  $V$  through an isentropic convergent nozzle of exit area  $A$ . The gas in the tank has an initial pressure  $p_i$  and an initial temperature  $T_i$ , and the ratio of atmospheric pressure to the final pressure in the tank is less than critical. Solve the problem, assuming that the gas in the tank undergoes one of the following two extreme cases of expansion:

- Isentropic.
- Isothermal.

**Solution**

(a) *Isentropic case.* When the blow-down process is extremely rapid or when the gas is thermally insulated, the process may be considered adiabatic. If, in addition, friction is negligible, the process is then considered isentropic. Assuming the gas to follow the perfect gas law, then:

$$pV = mRT$$

Differentiating with respect to time  $t$  gives:

$$V \frac{dp}{dt} = mR \frac{dT}{dt} + RT \frac{dm}{dt} \quad (a)$$

Expressions of  $dT/dt$  and  $dm/dt$  in terms of  $p$  and  $t$  are obtained as follows. For isentropic flow:

$$pV^\gamma = C \quad \text{or} \quad p^{1/\gamma} V = p^{1/\gamma} \frac{RT}{p} = C'$$

Hence:

$$p^{(1-\gamma)/\gamma} T = C''$$

By differentiating the logarithm of this equation, we have:

$$\frac{1 - \gamma}{\gamma} \frac{dp}{p} + \frac{dT}{T} = 0$$

and

$$\frac{dT}{dt} = \frac{\gamma - 1}{\gamma} \frac{T}{p} \frac{dp}{dt} \quad (b)$$

Since the atmospheric (back) pressure is less than the critical pressure, the Mach number at the exit of the nozzle is unity. The mass rate of flow, according to Eq. (3.24), is:

$$\frac{dm}{dt} = \frac{-pA}{\sqrt{T}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \quad (c)$$

where  $p$  and  $T$  are the pressure and temperature in the tank and  $A$  the area of the nozzle. Combination of Eqs. (b), (c), and (a) gives:

$$V \frac{dp}{dt} = mR \left( \frac{\gamma-1}{\gamma} \frac{T}{p} \frac{dp}{dt} \right) - RT \left[ \frac{pA}{\sqrt{T}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \right]$$

But since  $mRT/p = V$  and  $T = T_0(p/p_0)^{(\gamma-1)/\gamma}$ , therefore:

$$\begin{aligned} -V \frac{dp}{dt} \left( 1 - \frac{\gamma-1}{\gamma} \right) &= R\sqrt{T_0} \left( \frac{p}{p_0} \right)^{(\gamma-1)/2\gamma} pA \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \\ -\frac{dp}{dt} &= \left[ \frac{\gamma}{V} \frac{R\sqrt{T_0}A}{p_0^{(\gamma-1)/2\gamma}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \right] p^{(3\gamma-1)/2\gamma} \end{aligned} \quad (d)$$

or

$$\frac{dp}{dt} = -Cp^{(3\gamma-1)/2\gamma}$$

where:

$$C = \frac{\gamma}{V} \frac{R\sqrt{T_0}A}{p_0^{(\gamma-1)/2\gamma}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}}$$

Separating the variables and integrating Eq. (d) gives:

$$\int_{p_0}^{p'} p^{(1-3\gamma)/2\gamma} dp = -C \int_0^t dt$$

Therefore:

$$t = \frac{-1}{C} \frac{2\gamma}{1-\gamma} \left( p_0^{(1-\gamma)/2\gamma} - p'^{(1-\gamma)/2\gamma} \right) = \frac{-2\gamma}{C(1-\gamma)} p_0^{(1-\gamma)/2\gamma} \left[ \left( \frac{p'}{p_0} \right)^{(1-\gamma)/2\gamma} - 1 \right]$$

Substituting the value of  $C$  yields:

$$t = \frac{-2V}{\left( \frac{p'}{p_0} \right)^{(1-\gamma)/2\gamma} - 1} \left[ \left( \frac{p'}{p_0} \right)^{(1-\gamma)/2\gamma} - 1 \right] \quad (e)$$

For  $\gamma = 1.4$  and  $R = 287$  J/kg K, this equation reduces to:

$$t = \frac{0.43V}{A\sqrt{T_0}} \left[ \left( \frac{p'}{p_0} \right)^{-0.143} - 1 \right] \quad (f)$$

(b) *Isothermal case.* When the thermal capacity of the tank is much larger than that of the gas and the thermal resistance to heat transfer is negligible, a solution may be considered for the limiting case in which the temperature of the gas in the tank remains unchanged. In the following analysis the flow through the nozzle will be assumed isentropic while the gas in the tank expands isothermally. The perfect gas law is:

$$pV = mRT$$

Since  $T = \text{constant}$ , then:

$$\frac{dp}{dt} = \frac{dm}{dt} \frac{RT}{V}$$

Substituting for  $dm/dt$  from Eq. (3.24) gives:

$$\frac{dp}{dt} = \frac{-pA}{\sqrt{T}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \frac{RT}{V} = C'p$$

where:

$$C' = -\frac{A}{\sqrt{T}} \sqrt{\frac{\gamma}{R}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \frac{RT}{V}$$

Separating variables and integrating gives:

$$\int_{p_0}^{p'} \frac{dp}{p} = C' \int_0^t dt$$

or

$$t = \frac{1}{C'} \ln \frac{p'}{p_0}$$

Substituting the value of  $C'$  gives the following expression for  $t$ :

$$t = -\frac{1}{C'} \ln \frac{p'}{p_0} = -\frac{A\sqrt{\gamma RT}}{\sqrt{\gamma RT}} \sqrt{\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}} \ln \frac{p'}{p_0} \quad (g)$$

Substituting  $\gamma = 1.4$  and  $R = 287 \text{ J/kg K}$  gives:

$$t = \frac{-0.086V \ln \frac{P_t}{P_i}}{A\sqrt{T}^{1/2}} \quad (h)$$

From Eqs. (f) and (h) it can be shown that the time required to blow down a tank from an initial pressure to a final pressure is greater in the isothermal than in the isentropic case. This is also indicated in Fig. 3.9, which shows the pressure-time history for the two cases considered. In the isentropic case no heat interaction takes place, whereas in the isothermal case maximum heat interaction occurs in order to maintain the temperature constant. Since these cases are the two possible extremes, it is reasonable to expect the actual blow-down time to lie between them. Note that the experimental data points shown in Fig. 3.9 are initially nearer to the isentropic curve but gradually shift toward the isothermal curve as time increases. This is consistent with the fact that during the first few seconds practically no heat transfer takes place, owing to the rapidity of the process. As the temperature of the gas drops, heat is transferred from the tank wall to the gas, and this is the reason for the shift toward the isothermal curve. A computer program for solving Eqs. (f) and (g) is shown in Table 3.2; the flow chart is shown in Fig. 3.10.

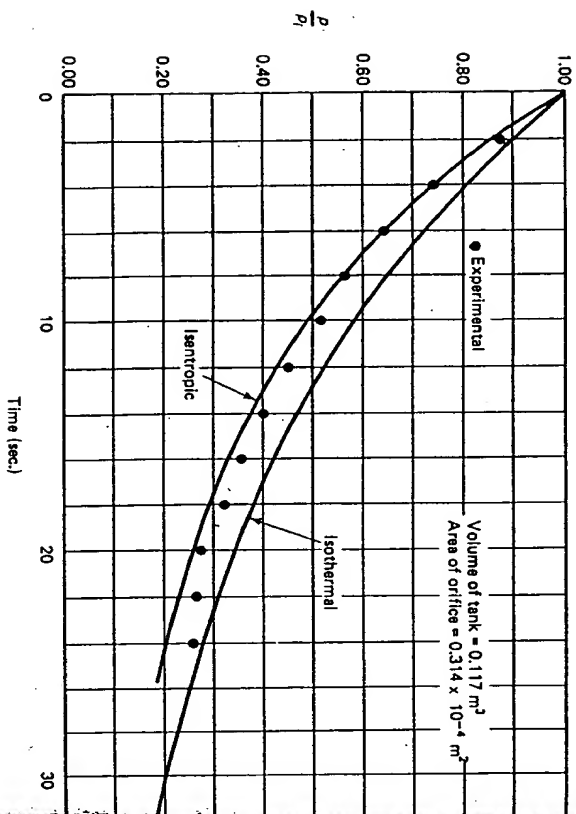


Figure 3.9 Pressure-time history during the blow-down of a pressurized tank through an orifice.

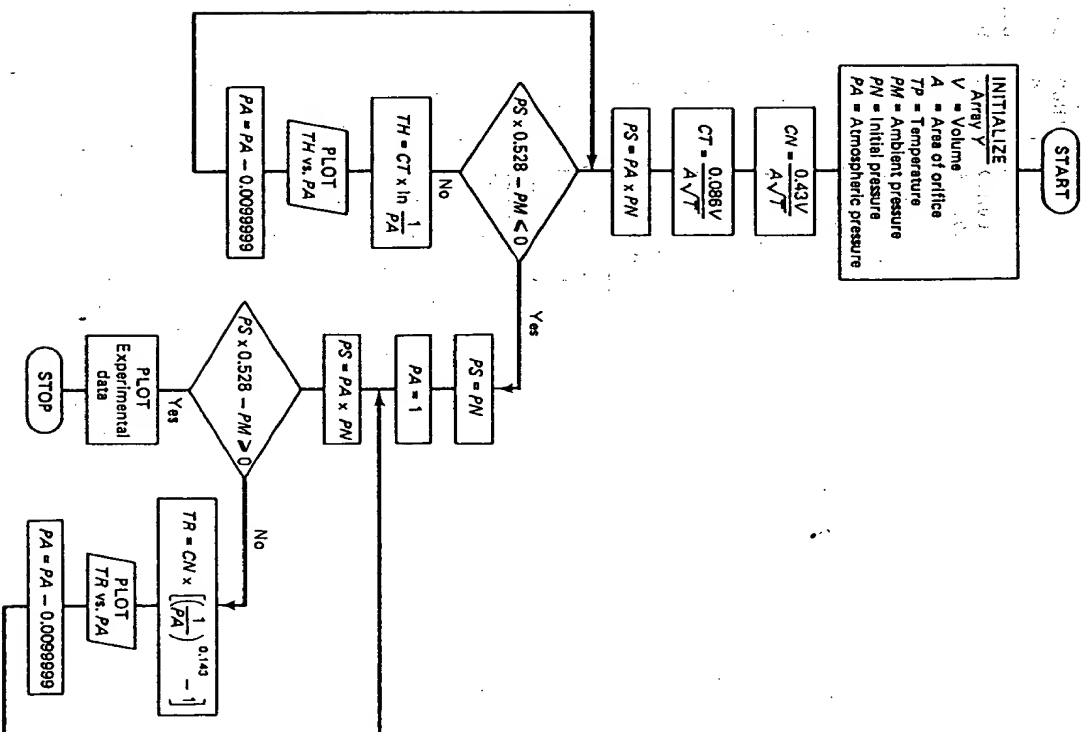


Figure 3.10 Flow chart for Example 3.3.

TABLE 3.2 COMPUTER PROGRAM OF PRESSURE-TIME HISTORY DURING A BLOW-DOWN PROCESS

```

PROGRAM ISD
DIMENSION Y(12)
DATA
Y/138.,115.5,100.5,87.,79.5,69.5,62.,54.5,49.5,42.,40.5,40./
DATA AREA/ 3.1412E-05/, VOL/.1167/
C* AREA = AREA OF ORIFICE IN SQUARE METERS*
C* VOL = VOLUME OF TANK, IN CUBIC METERS*
DATA TMPK/22./, DIAM/.072/, PRMB/1.013E05/, PRIN/9.61E05/
C*TMPK IS TEMPERATURE IN DEGREES CELSIUS.
C*DIAM IS DIAMETER AT THROAT IN METERS.
C*PRMB IS AMBIENT PRESSURE IN NEWTONS PER METER SQUARE.
C*PRIN IS INITIAL PRESSURE IN NEWTONS PER METER SQUARE.
CALL SCALF(.25, 5., 0., 0.)
CALL FGRID(0, 0., 0., 10., 3)
CALL FGRID(1, 0., 0., .2, 5)
C**NUMBERING THE X-AXIS
R = -.4
DO 25 I=1,4
J = R + .40001
CALL FCHAR (R, -.06, .1, .2, 0)
WRITE (6, 22) J
22 FORMAT (12)
R = R + 10.
25 CONTINUE
C**NUMBERING THE Y-AXIS
S = -.02
DO 35 I=1,6
T = S + .02
CALL FCHAR (-2., 5, .1, .2, 0)
WRITE (6, 32) T
32 FORMAT (F4.2)
S = S + .20001
35 CONTINUE
C**HEADING
CALL FCHAR(10.8, -.14, .2, .2, 0)
WRITE (6, 40)
40 FORMAT (11H TIME (SEC.))
CALL FCHAR (-5.2, .88, .2, .2, 0)
WRITE (6, 50)
50 FORMAT (3HP/P)
CALL FCHAR (-2.8, .88, .1, .1, 0)
WRITE (6, 55)
55 FORMAT (1H0)
C**LABELS
CALL FCHAR (6., .8, .1, .1, 0)
WRITE (6, 60)
60 FORMAT (16H0 = EXPERIMENTAL)
CALL FCHAR (11.2, .3, .1, .1, 0)
WRITE (6, 61)

```

TABLE 3.2 COMPUTER PROGRAM OF PRESSURE-TIME HISTORY DURING A BLOW-DOWN PROCESS (Continued)

```

61 FORMAT (10HISENTROPIC)
CALL FCHAR (18., .4, .1, .1, 0)
WRITE (6, 62)
62 FORMAT (10HISOTHERMAL)
C*START THE CALCULATIONS
PRIN = PRIN + PRMB
TMPK = 273.15 + TMPK
CNST = VOL/AREA/(TMPK)**.5
CISN = .43 * CNST
CIST = .086 * CNST
C**PLOTING ISOTHERMAL
CALL FPLT (-2, 0., 1.)
PRES = PRIN
PRAT = PRES / PRIN
412 PRES = PRAT * PRIN
IF (PRES *.528 - PRMB) 601, 601, 314
314 ZTH = ALOG (1./PRAT)
TMTH = CIST * ZTH
CALL FPLT (0, TMTH, PRAT)
PRAT = PRAT - .0099999
GO TO 412
C**PLOTING ISENTROPIC
601 PRES = PRIN
CALL FPLT (1, 0., 1.)
CALL FPLT (2, 0., 1.)
PRAT = PRES / PRIN
612 PRES = PRAT * PRIN
IF (PRES *.528 - PRMB) 501, 501, 414
414 ZTR = (1. / PRAT)**.143
TMTR = CISN * (ZTR - 1.)
CALL FPLT (0, TMTR, PRAT)
PRAT = PRAT - .0099999
GO TO 612
C**PLOTING EXPERIMENTAL
501 Q = 2.
DO 210 I=1,12
P = Q**1.2
R = Y(I) / PRIN - .01
DO 209 K=1,3
CALL FCHAR (P, R, .1, .1, 0)
WRITE (6, 206)
206 FORMAT (1H0)
209 CONTINUE
Q = Q + 2.
210 CONTINUE
STOP
END
END*

```

## 3.5 ISENTROPIC FLOW THROUGH A NOZZLE

The compressibility of a gas affects the flow properties of the gas when it is flowing at high speeds. Two cases will be considered—the convergent nozzle and the convergent-divergent nozzle.

(a) *Convergent Nozzle.* Consider the flow of a perfect gas through a convergent nozzle, as shown in Fig. 3.11. The nozzle discharges into a plenum chamber, in which the pressure  $p_b$  can be regulated. Let  $p_e$  be the exit pressure just inside the nozzle and subscript 0 denote stagnation conditions. When  $p_b$  is reduced below  $p_0$ , gas is drawn through the nozzle. As  $p_b$  is reduced, the mass rate of flow increases monotonically until no further increase in rate of mass flow is noted, regardless of any further decrease in  $p_b$ . This pressure, which corresponds to the maximum rate of flow, is the critical pressure and is shown as  $p^*$  in Fig. 3.11. At pressures between  $p_1$  and  $p^*$ , the exit pressure equals the back pressure<sup>†</sup> and the flow in the nozzle is able to sense changes in back pressure. If the back pressure is lower than the exit pressure (such as at  $p_4$ ), the critical pressure and the conditions upstream of the nozzle exit as well as the mass rate of flow are not affected. This corresponds to a state at which the nozzle is said to be *choked*, and changes in the back pressure are not sensed upstream of the nozzle exit. An irreversible balance between the exit pressure and the back pressure occurs discontinuously by lateral expansion of the stream from  $p_e$  to  $p_b$  outside the nozzle.

Now let us investigate the value of the exit Mach number when gas is flowing through the nozzle at a maximum rate. According to Eq. (3.24) the maximum mass flux  $G$  (mass rate of flow per unit area) occurs at  $M = 1$ . Therefore, if  $p_e$  is equal to  $p^*$ , which occurs when the back pressure is equal to or less than  $p^*$ , flow at the exit is sonic and the mass rate of flow through the nozzle is a maximum. If the back pressure exceeds  $p^*$ , flow everywhere is subsonic and the mass rate of flow is less than its maximum (unchoked). Figure 3.11 shows the mass rate of flow as a function of  $p/p_0$ . Also shown in the same figure is a plot of  $p_e/p_0$  versus  $p_b/p_0$ .

When the flow is choked, the nozzle acts as a flow-metering device. The mass rate of flow depends only on stagnation temperature, stagnation pressure, and the nozzle exit area and remains constant for all back pressures less than or equal to  $p^*$ .

(b) *Convergent-Divergent Nozzle.* Consider the flow of a perfect gas in a convergent-divergent nozzle, as shown in Fig. 3.12. In this case, the exit pressure can be less than the critical value in the diverging portion of the nozzle. When  $p_b = p_0$ , there is no flow in the nozzle; when the back pressure is decreased below  $p_0$ , gas is drawn through the nozzle, its static pressure decreasing in the convergent part of the nozzle and reaching a minimum at the throat, then in-

<sup>†</sup> If the back pressure is less than the exit pressure and the Mach number at the exit is less than 1, the fluid expands laterally upon leaving the nozzle. This causes a decrease in velocity and a corresponding increase in pressure. Therefore,  $p_e$  will never adjust to the lower back pressure. Hence,  $p_e$  can never exceed  $p_b$  unless the latter is less than  $p^*$ .

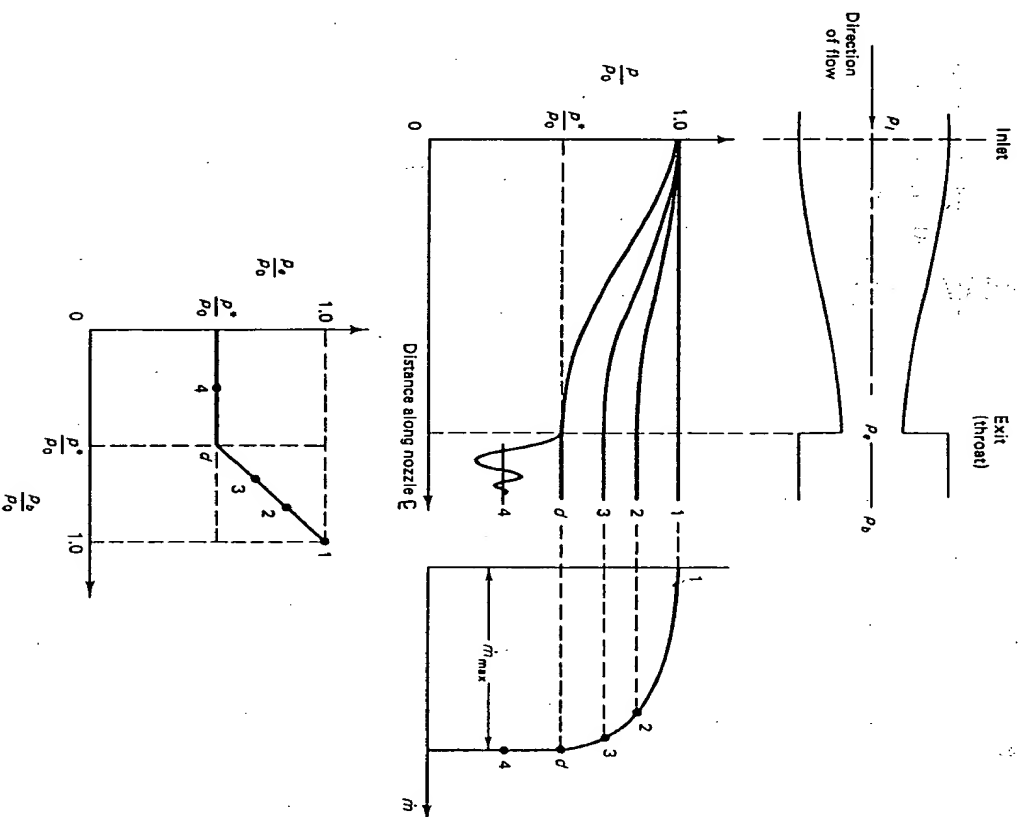


Figure 3.11 Effect of back pressure on the flow in a convergent nozzle.

creasing in the divergent part. At the same time, the velocity increases until it is at a maximum at the throat of the nozzle, and then the velocity decreases in the diverging portion. The flow is similar to that in a conventional venturi, where the converging portion of the duct acts as a nozzle while the diverging portion acts as a diffuser. When the fluid accelerates in the convergent part of the nozzle, the velocity increases at a faster rate than the density decreases, so that the mass flux  $G$  increases. As the back pressure is further decreased, the mass rate of flow increases; however, below a certain back pressure the mass flow rate does not

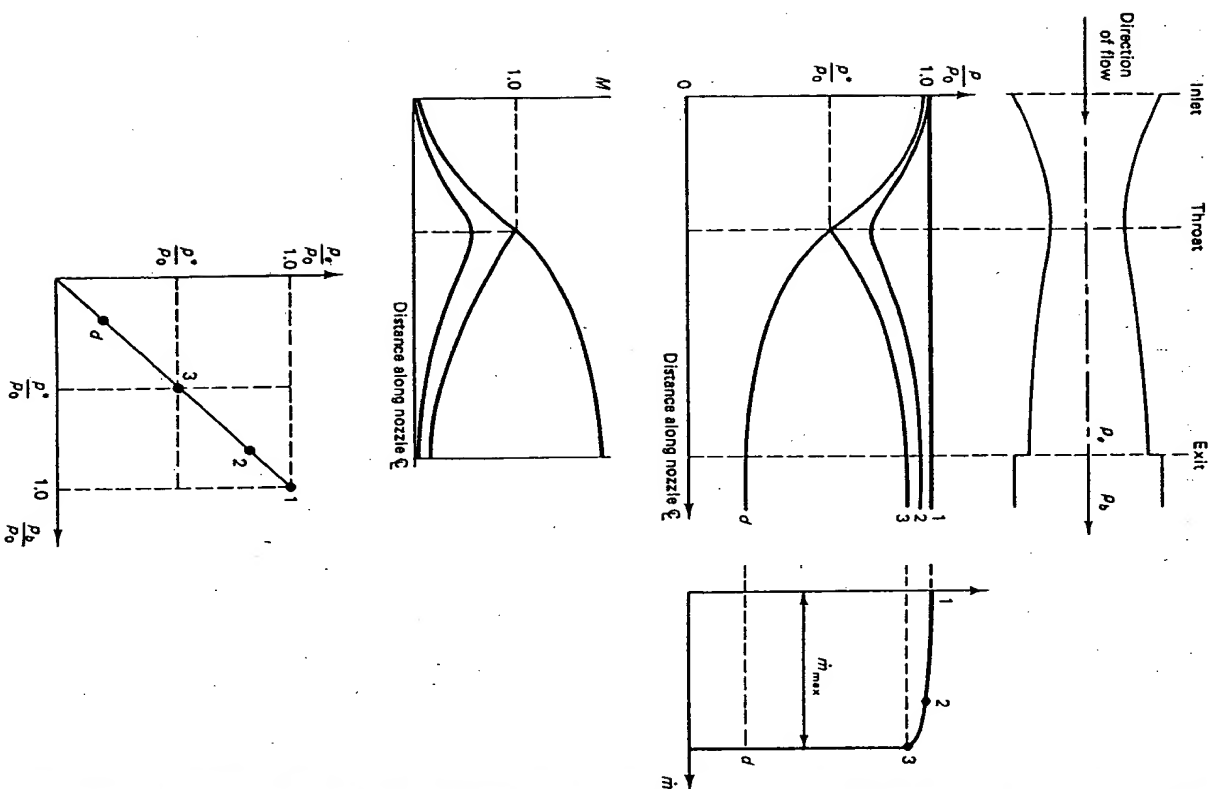


Figure 3.12 Isentropic flow through a convergent-divergent nozzle.

change, even though the back pressure is reduced. This limiting case represents choked flow, corresponding to sonic flow at the throat ( $M = 1$ ) and maximum value of  $G$ .

If the back pressure is low enough, the flow continues to accelerate, after reaching sonic velocity at the throat, and supersonic speeds prevail in the diverging part of the nozzle. As the fluid accelerates in supersonic flow, the density decreases at a faster rate than the velocity increases, so that the mass flux  $G$  decreases.

The nozzle behaves like a subsonic venturi if the back pressure is equal to  $p_3$  or higher; on the other hand, it is a supersonic convergent-divergent nozzle if the back pressure is the same as the design pressure,  $p_d$ . What happens if the back pressure is between  $p_3$  and  $p_d$  or below  $p_d$  will be discussed in Chapter 4. But for the present it may be stated that flow without losses (reversible flow) cannot be attained if the back pressure is between  $p_3$  and  $p_d$  or if the back pressure is below  $p_d$ .

The mass rate of flow through a nozzle is limited by the throat area. Since the area is a minimum at the throat, the mass rate of flow per unit area has a maximum value at the throat. Figure 3.12 shows the variation of flow rate,  $\dot{m}$ , as a function of pressure ratio,  $p/p_0$ . Also shown is exit pressure,  $p_d/p_0$ , as a function of back pressure,  $p/p_0$ .

When a fluid flows through a passage of variable cross section, the maximum rate of flow per unit area occurs at the section where the Mach number is unity. This section is often at the exit of a converging nozzle or at the throat of a convergent-divergent nozzle. For each duct there is a minimum area ratio,  $A^*/A$ , corresponding to choked flow. A reduction of area below this area ratio results merely in a reduction of the mass rate of flow, but a Mach number of unity is still maintained at the minimum section. The maximum mass flux corresponding to choked conditions is given by Eq. (3.24), while the corresponding critical properties are given by Eqs. (3.8) to (3.11).

The properties of a fluid at any point in a nozzle can be calculated from the equations of Secs. 3.2 and 3.3. From these equations it is possible to determine whether a parameter will increase or decrease in value in traveling downstream. Table 3.3 indicates, with + and - symbols, these changes. Figures 3.13 and 3.14 summarize the properties of fluids in isentropic flow.

TABLE 3.3

	$dA$	$dM$	$dV$	$dp$	$dT$	$dp$
<i>Subsonic flow (<math>M &lt; 1</math>):</i>						
Converging nozzle	-	+	+	-	-	-
Diverging diffuser	+	-	-	+	+	+
<i>Supersonic flow (<math>M &gt; 1</math>):</i>						
Converging diffuser	-	-	-	+	+	+
Diverging nozzle	+	+	+	-	-	-

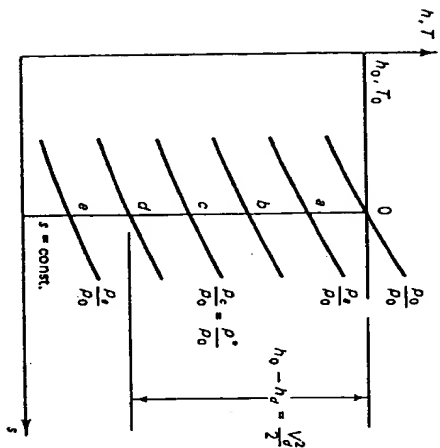
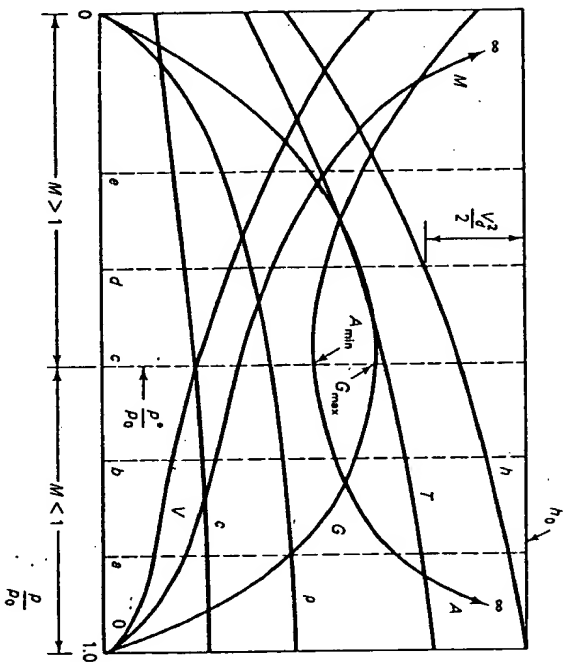


Figure 3.13 Isentropic flow.

Figure 3.14 Flow properties versus  $p/p_0$  for isentropic flow (not to scale).

## Example 3.4

Air at 403 K and 1 atm enters a convergent nozzle at a velocity of 150 m/s and expands isentropically to an exit pressure of 76 kPa. If the inlet area of the nozzle is  $5 \times 10^{-3} \text{ m}^2$ , find:

- The stagnation temperature, pressure, and enthalpy.
- The Mach number at the inlet.
- The temperature, Mach number, and area at the exit.
- What must be the back pressure, temperature, and flow rate if sonic conditions are attained at the exit?

Assume air to be a perfect gas having  $\gamma = 1.4$ .

## Solution

- (a) From Eq. (1.32), the stagnation temperature is:

$$T_0 = T_1 + \frac{V_1^2}{2c_p} = 403 + \frac{(150)^2}{2 \times 10^3} = 403 + 11.25 = 414.25 \text{ K}$$

where subscript 1 indicates conditions at the inlet. Using the isentropic relation:

$$\frac{p_0}{p_1} = \left(\frac{T_0}{T_1}\right)^{\gamma/(\gamma-1)}$$

the stagnation pressure  $p_0$  is therefore:

$$p_0 = 101.3 \left(\frac{414.25}{403}\right)^{1.4/(1.4-1)} = 111.56 \text{ kPa}$$

The stagnation enthalpy is:

$$h_0 = h_1 + \frac{V_1^2}{2} = c_p T_0 = 1.0 \times 414.25 = 414.25 \text{ kJ/kg}$$

$$(b) M_1 = \frac{V_1}{\sqrt{\gamma R T_1}} = \frac{150}{20.1\sqrt{403}} = 0.372$$

- (c) The conditions at the exit are:

$$T_2 = T_0 \left(\frac{p_2}{p_0}\right)^{(\gamma-1)/\gamma} = 414.25 \left(\frac{76}{111.56}\right)^{0.286} = 371.18 \text{ K}$$

$$V_2 = \sqrt{2(h_0 - h_2)} = \sqrt{2 \times 10^3(414.25 - 371.18)} = 293.5 \text{ m/s}$$

$$M_2 = \frac{V_2}{20.1\sqrt{T_2}} = \frac{293.5}{20.1\sqrt{371.18}} = 0.758$$

The mass rate of flow is:

$$\dot{m} = \rho_1 A_1 V_1 = \left( \frac{1.013 \times 10^5}{287 \times 403} \right) (5 \times 10^{-3})(150) = 0.657 \text{ kg/s}$$

Therefore:

$$A_2 = \frac{\dot{m}}{\rho_2 V_2} = \frac{0.657}{\left( \frac{0.76 \times 10^5}{287 \times 371.18} \right) (293.5)} = 31.38 \times 10^{-4} \text{ m}^2 = 31.38 \text{ cm}^2$$

- (d) For maximum flow rate  $M_2 = 1$  at exit and  $V_2 = c_2$ . The critical pressure and temperature, according to Eqs. (3.9) and (3.8), are:

$$p^* = 0.528 p_0 = 0.528 \times 111.56 = 58.9 \text{ kPa}$$

$$T^* = 0.8333 T_0 = 0.8333 \times 414.25 = 345.2 \text{ K}$$

The mass rate of flow is:

$$\dot{m} = \rho^* A^* V^* = \left( \frac{0.589 \times 10^3}{287 \times 345.2} \right) (31.38 \times 10^{-4})(20.1\sqrt{345.2}) = 0.697 \text{ kg/s}$$

### Example 3.5

Air at a temperature of 284 K and atmospheric pressure flows isentropically through a convergent-divergent nozzle. The velocity at the inlet is 150 m/s and the inlet area is 10 cm<sup>2</sup>. If the flow at the exit of the nozzle is supersonic, find:

- The Mach number at the inlet.
- Stagnation temperature and pressure.
- The temperature and pressure at the throat.
- The velocity and Mach number at the exit if  $T_2 = 220 \text{ K}$ .
- The area at the throat.

Assume air to be a perfect gas of  $\gamma = 1.4$ .

### Solution

- (a) The Mach number at the inlet is:

$$M_1 = \frac{V_1}{c_1} = \frac{150}{20.1\sqrt{284}} = 0.443$$

- (b) The stagnation temperature and pressure, according to Eqs. (3.4) and (3.5), are:

$$T_0 = T_1 \left( 1 + \frac{\gamma-1}{2} M_1^2 \right) = 284 \left[ 1 + \frac{0.4}{2} \times (0.443)^2 \right] = 295 \text{ K}$$

$$p_0 = p_1 \left( 1 + \frac{\gamma-1}{2} M_1^2 \right)^{\gamma/(\gamma-1)} = 101.3(1.0393)^{3.5} = 115.93 \text{ kPa}$$

- (c) Since the flow at the exit is supersonic, the Mach number at the throat of the nozzle must be equal to 1. Therefore:

$$\frac{T^*}{T_0} = 0.8333 \quad \text{and so} \quad T^* = 245.82 \text{ K}$$

$$\frac{p^*}{p_0} = 0.5283 \quad \text{and so} \quad p^* = 61.25 \text{ kPa}$$

- (d) The mass rate of flow is:

$$\begin{aligned} \dot{m} &= \rho_1 A_1 V_1 = \frac{p_1}{R T_1} A_1 V_1 \\ &= \left( \frac{1.013 \times 10^5}{287 \times 284} \right) (10 \times 10^{-4})(150) = 0.186 \text{ kg/s} \end{aligned}$$

The Mach number at the exit may be obtained from the relation:

$$\frac{T_0}{T_2} = 1 + \frac{\gamma-1}{2} M_2^2$$

from which:

$$M_2^2 = \frac{2}{\gamma-1} \left( \frac{T_0}{T_2} - 1 \right) = \frac{2}{0.4} \left( \frac{295}{220} - 1 \right) = 1.70$$

and

$$M_2 = 1.30$$

Therefore:

$$V_2 = M_2 c_2 = 1.3 \times 20.1\sqrt{220} = 387.57 \text{ m/s}$$

- (e) The area at the throat is:

$$\begin{aligned} A_t &= \frac{\dot{m}}{\rho_t V_t} \\ &= \frac{0.186}{\left( \frac{0.6125 \times 10^5}{287 \times 245.82} \right) (20.1\sqrt{245.82})} = 6.798 \times 10^{-4} \text{ m}^2 = 6.798 \text{ cm}^2 \end{aligned}$$

### 3.6 TABULAR AND GRAPHICAL REPRESENTATION OF ISENTROPIC RELATIONS

From relations derived in the previous sections, values of properties have been calculated as a function of Mach number. These are tabulated in *Gas Tables*<sup>\*</sup> for different values of  $\gamma$ . Properties are listed in these tables in a nondimensional

<sup>\*</sup> Refer to Keenan and Kaye, *Gas Tables*, or NACA report 1135. An abstract of similar tables is given in Table A2 in the Appendix.

form referred to the corresponding property at either the stagnation point or the Mach 1 point. The chart shown in Fig. 3.15† presents the equations of flow in graphical form. The chart presents properties of a perfect gas of  $\gamma = 1.4$  when flowing adiabatically. Because of reading errors, data obtained from this chart are not as accurate as those from *Gas Tables*. Nonetheless, the relationships between various properties and pressure, and also the representation of flow processes, are more evident in the graph than in the tables.

The ordinates of the chart (Fig. 3.15) are mass flux and area ratios ( $G/G^*$ ), ( $A^*/A$ ), while the abscissa is pressure ratio,  $p/p_0$ , where  $p_0$  is the initial stagnation pressure. Plotted on this chart are lines of constant entropy. These are called isentropes. The outer curve is the *primary isentrope*; the inner curves, which appear dotted, are *secondary isentropes*. Reversible adiabatic processes can be described by a single isentrope; also, the stagnation pressure is constant along each isentrope. If an irreversible process occurs, there is a change in stagnation pressure and the stagnation pressure then is identified by subscript 02. The stagnation pressure of a secondary isentrope, referred to the stagnation pressure of the primary isentrope, is indicated by the ratio  $p_{02}/p_{01}$ . When the stagnation pressure diminishes, there is a corresponding increase in entropy.

Lines of constant Mach number are also shown on the chart. These originate at the lower right-hand corner of the chart and extend to the primary isentrope. On the line marked  $M = 1$  the change of entropy expressed on a dimensionless basis as  $\Delta s/R$  is shown relative to the primary isentrope. The region at the left of the line  $M = 1$  applies to subsonic flow, while the region at the right represents supersonic flow. The flow across a normal shock wave is indicated by the loop shown on the chart. The primary isentrope to the right of the sonic line represents the supersonic portion of the loop. This part of the curve indicates gas properties before the shock ( $M > 1$ ), while the left curve of the loop shows properties after the shock ( $M < 1$ ). Normal shock waves will be discussed in more detail in Chapter 4.

Also shown on the chart are scales relating to  $M$  (Mach number),  $4fL^*/D_H$  (friction-factor ratio),  $V/c_0$  (velocity ratio), and  $T/T_0$  (temperature ratio). The parameter  $4fL^*/D_H$ , which applies to friction effects in a constant-area duct, will be discussed in Chapter 5.

This section describes how this chart was constructed. As shown schematically in Fig. 3.16, the primary isentrope is plotted from values of  $G/G^*$  or  $A^*/A$  and  $p/p_0$ , obtained from Eqs. (3.26) and (3.5) for various selected values of Mach number. In plotting secondary isentropes it should be noted that  $T_0$  remains constant across an irreversibility, assuming that there are no heat or work interactions with the environment. Therefore, the maximum mass fluxes along an irreversible process, according to Eq. (3.24), are related to the stagnation pressures:

$$\frac{G_1^*}{G_2^*} = \frac{p_{02}}{p_{01}} \quad (3.29)$$

† See reference 10.

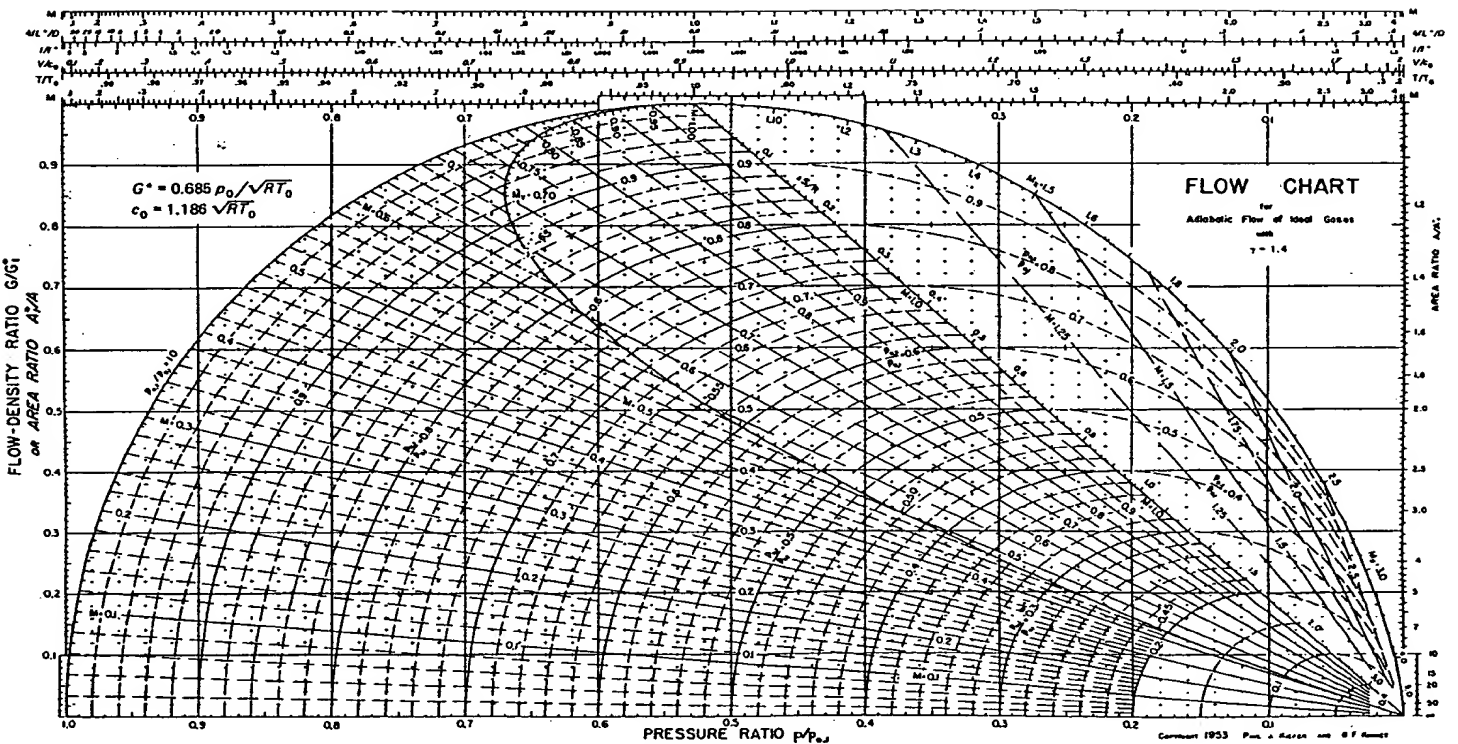


Figure 3.15 Flow chart for adiabatic flow of ideal gases with  $\gamma = 1.4$ . (Copyright 1953, Paul J. Kieffer and G. F. Kinney.)

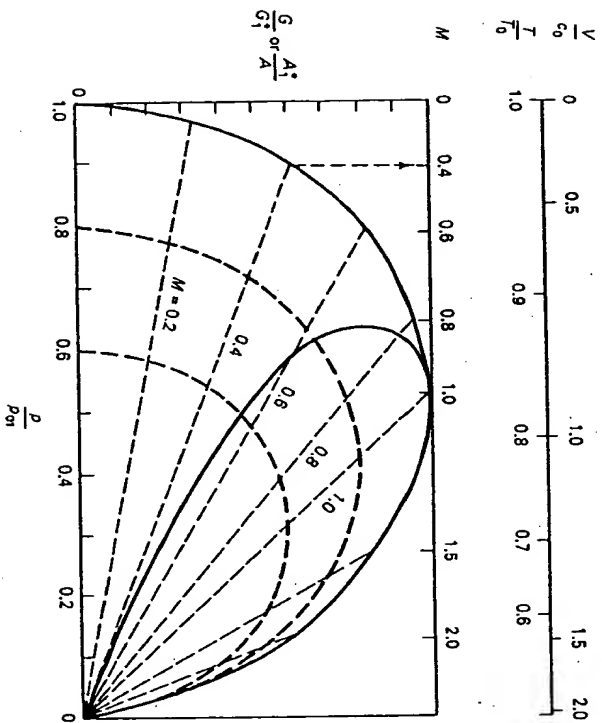


Figure 3.16 Flow chart.

According to Eq. (3.26), mass flux  $G$  at a given Mach number is proportional to the maximum mass flux ( $G^*$ ):

$$\frac{G_2}{G_1^*} = \frac{G_2}{G_1} = \frac{G_2}{G_1^*} \quad \frac{G_2}{G_1^*} = \frac{G_2}{G_1} = \frac{G_2}{G_1^*}$$

The stagnation pressures can therefore be expressed as:

$$\frac{G_2}{G_1^*} = \frac{p_{02}}{p_{01}}, \quad \frac{G_2}{G_1^*} = \frac{p_{02}}{p_{01}}$$

But pressure at a given Mach number is proportional to stagnation pressure:

$$\frac{p_{02}}{p_{01}} = \frac{p_2}{p_1} = \frac{p_2}{p_1}$$

Therefore mass fluxes are related to pressures as follows:

$$\frac{G_2}{G_1^*} = \frac{p_2}{p_{01}}, \quad \frac{G_2}{G_1^*} = \frac{p_2}{p_{01}} \quad (3.30)$$

Accordingly, points 1 and 2 lie on the same Mach line; similarly, on any one Mach line values of  $p_{02}/p_{01}$  must conform to the relationship:

$$\frac{G_2}{G_1^*} = \frac{p_2}{p_{01}} = \frac{p_{02}}{p_{01}}, \quad \frac{G_2}{G_1^*} = \frac{p_2}{p_{01}} = \frac{p_{02}}{p_{01}}$$

The state of the flow at any section in a system is represented as a point which is located on the chart from the values of such parameters as  $M$ ,  $p/p_{01}$ ,  $T/T_0$ , and  $A/A_1^*$ . Use of this flow chart is illustrated by applying it to Examples 3.4 and 3.5.

In Example 3.4, the flow is isentropic and therefore reference need be made only to the primary isentrope. The Mach number at the inlet is as follows:

$$M_1 = \frac{150}{20.1\sqrt{T_1}} = 0.372$$

As shown in Fig. 3.17, at  $M_1 = 0.372$ , the following data are obtained from the chart:

$$\frac{p_1}{p_{01}} = 0.904, \quad \text{from which } p_{01} = 112.06 \text{ kPa}$$

$$\frac{T_1}{T_0} = 0.973, \quad \text{from which } T_0 = 414.2 \text{ K}$$

$$\frac{A_1^*}{A_1} = 0.595, \quad \text{from which } A_1^* = 29.75 \times 10^{-4} \text{ m}^2$$

The stagnation enthalpy can then be calculated:

$$h_0 = c_p T_0 = 1.0 \times 414.2 = 414.2 \text{ kJ/kg}$$

At point 2, the pressure is 76 kPa and the pressure ratio is:

$$\frac{p_2}{p_{01}} = \frac{76}{112.06} = 0.678$$

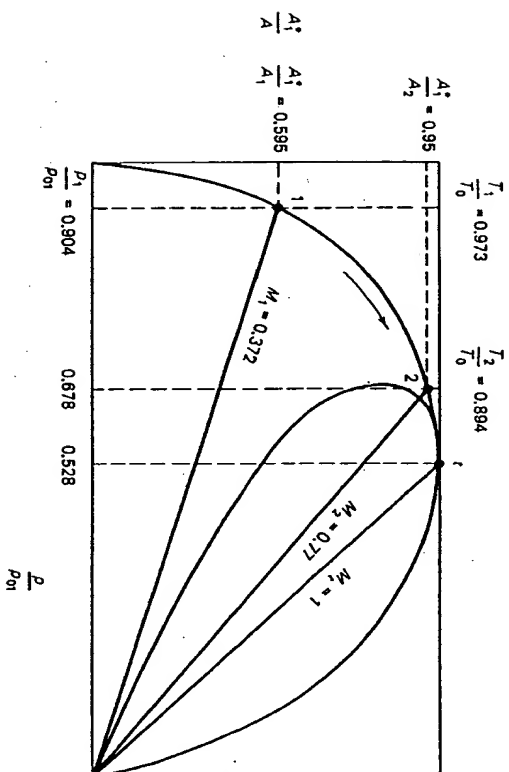


Figure 3.17

At point 2 on the primary isentrope, the following values are obtained:

$$M_2 = 0.770$$

$$\frac{T_2}{T_0} = 0.894, \quad \text{from which } T_2 = 370.3 \text{ K}$$

$$\frac{A^*}{A_2} = 0.950, \quad \text{from which } A_2 = 31.32 \times 10^{-4} \text{ m}^2$$

If the Mach number at the exit is unity, pressure and temperature values at the exit are therefore:

$$\frac{P^*}{P_{01}} = 0.528, \quad P^* = 59.17 \text{ kPa}$$

and

$$\frac{T^*}{T_0} = 0.833, \quad T^* = 345 \text{ K}$$

The mass rate of flow is then calculated as in Example 3.4. The solution of Example 3.5 appears in Fig. 3.18.

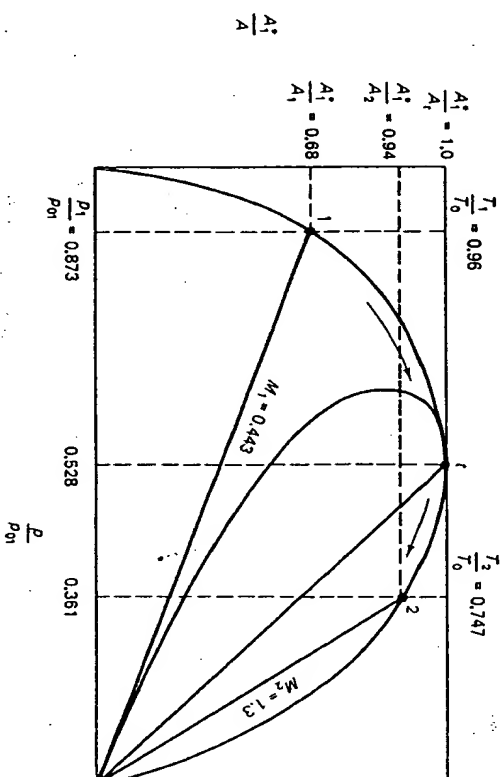


Figure 3.18

### 3.7 IMPULSE FUNCTION

When a fluid flows through a duct, the fluid exerts force on all points of the inside walls of the solid boundaries. These forces arise from both pressure and shear effects as well as momentum effects. In a rocket engine, the resultant of these forces is a net force which leads to propulsion. The direction of thrust is opposite to the direction of gas flow, and if the thrust exceeds external drag forces, the system will accelerate; if the thrust is less than external drag forces, the system will decelerate.

The thrust developed by a fluid as it flows between two sections in a duct may be expressed as the difference in the impulse function at these two sections. As shown in Fig. 3.19, the forces acting on the control volume in the x-direction

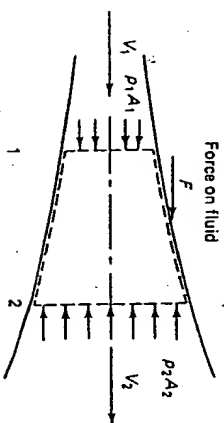


Figure 3.19 Forces acting on a control volume.

arise from pressure effects and from reaction to the thrust. Considering forces acting on the control volume shown, the momentum equation for steady flow is:

$$F + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1$$

where  $F$  is the wall force exerted on the fluid by the inner walls of the duct in the direction of flow. Note that the force  $F$  acts in a direction opposite to thrust.

The thrust produced by the stream between sections 1 and 2 is:

$$F = (p_2 A_2 + \dot{m} V_2) - (p_1 A_1 + \dot{m} V_1) = I_2 - I_1 \quad (3.31)$$

where  $I$ , the impulse function, is defined as:

$$I = pA + \dot{m}V = pA \left( 1 + \frac{\rho V^2}{p} \right) \quad (3.32)$$

Note that Eq. (3.31) applies to flow with friction, heat transfer, area change, and so on. In order to relate impulse function with Mach number, the velocity of a perfect gas is first expressed in terms of Mach number as:

$$V^2 = c^2 M^2 = \gamma R T M^2 = \gamma \frac{p}{\rho} M^2$$

Therefore, impulse function is:

$$I = pA(1 + \gamma M^2) \quad (3.33)$$

The thrust developed per unit mass rate of flow defines the specific impulse:

$$I_s = \frac{F}{\dot{m}} \quad (3.34)$$

The units are N s/kg. In a typical rocket motor, there is no mass flow into the rocket. Applying the momentum equation to a control volume surrounding the rocket, the thrust is given by:

$$F = \dot{m} V_e + (p_e - p_{\text{atm}}) A_e$$

But it has been shown earlier in this chapter that the velocity of a gas flowing out of an isentropic nozzle can be expressed in terms of pressure, temperature, and  $\gamma$ :

$$V_e = \sqrt{\frac{2\gamma}{\gamma-1} R T_0 \left[ 1 - \left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} \right]} \quad (3.35)$$

Hence, the thrust can be similarly expressed:

$$F = \dot{m} \sqrt{\frac{2\gamma}{\gamma-1} R T_0 \left[ 1 - \left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} \right]} + (p_e - p_{\text{atm}}) A_e \quad (3.36)$$

It is useful to determine the area at the exit which provides maximum thrust.

The partial derivative of thrust with respect to area is set = 0 to maximize the thrust so that:

$$\left( \frac{\partial F}{\partial A_e} \right)_{p_e} = p_e - p_{\text{atm}} = 0$$

Obviously, then, the exit pressure should be the same as the ambient pressure:

$$p_e = p_{\text{atm}} \quad (3.37)$$

Similarly, the partial derivative of thrust with respect to pressure is set = 0 to maximize the thrust so that:

$$\left( \frac{\partial F}{\partial p_e} \right)_{A_e} = 0$$

which gives the following expression for the exit area:

$$A_e = \frac{\dot{m} R T_0}{\sqrt{\frac{2\gamma}{\gamma-1} R T_0 \left[ 1 - \left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} \right]}} \left[ \frac{p_e^{1/\gamma}}{\rho_0^{(\gamma-1)/\gamma}} \right] \quad (3.38)$$

This equation indicates that at any specified area ratio there is a particular exit pressure which is associated with maximum thrust.

When the exit pressure and the ambient pressure are identical, the corresponding thrust, which is called the optimum thrust, is:

$$F_{\text{optimum}} = \dot{m} \sqrt{\frac{2\gamma}{\gamma-1} R T_0 \left[ 1 - \left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} \right]} \quad (3.39)$$

The optimum specific impulse is therefore:

$$(I_s)_{\text{optimum}} = \sqrt{\frac{2\gamma R T_0}{\gamma-1} \left[ 1 - \left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} \right]} \quad (3.40)$$

On the other hand, there is an exit pressure which is associated with the maximum specific impulse and is of such value that:

$$\left( \frac{p_e}{p_0} \right)^{(\gamma-1)/\gamma} = 0 \quad \text{or} \quad p_e = 0$$

This maximum impulse is therefore obtained at zero exit pressure and is calculated from:

$$(I_s)_{\text{max}} = \sqrt{\frac{2\gamma R T_0}{\gamma-1}} \quad (3.41)$$

An exit pressure of zero cannot be obtained, however, unless the area ratio is infinitely large. Note that the conditions for the optimum impulse and the max-

imum impulse are different. In the former the flow expands isentropically in the nozzle so that the exit pressure is equal to the back pressure, whereas in the latter higher exit velocity is attainable (and consequently higher thrust), as the flow expands to zero pressure (at least theoretically) at the exit of the nozzle.

The effectiveness of a propulsion system can be rated by means of the impulse function. At Mach 1, the impulse function of a rocket is:

$$I^* = p^* A^* (1 + \gamma)$$

The impulse function at the exit plane, normalized with reference to that at the critical state, is therefore:

$$\frac{I}{I^*} = \frac{p}{p^*} \frac{A}{A^*} \frac{1 + \gamma M^2}{1 + \gamma}$$

Since pressure and area ratios can also be expressed as function of Mach number and  $\gamma$ , Eqs. (3.13) and (3.26), the impulse function ratio is therefore:

$$\frac{I}{I^*} = \frac{1 + \gamma M^2}{M \sqrt{2(\gamma + 1) \left(1 + \frac{\gamma - 1}{2} M^2\right)}} \quad (3.42)$$

Values of  $I/I^*$  are tabulated as a function of  $M$  and  $\gamma$  in Table A2 in the Appendix. In Fig. 3.5,  $I/I^*$  is plotted versus  $M$  for  $\gamma = 1.4$ .

### Example 3.6

Find the force produced by 2 kg/s of air on the inner walls of the convergent nozzle shown in Fig. 3.20. Note that the flow is not isentropic, as can be ascertained by calculating the entropy change from the temperatures and pressures at the inlet and exit of the nozzle (see Problem 3.24). Assume steady one-dimensional flow.

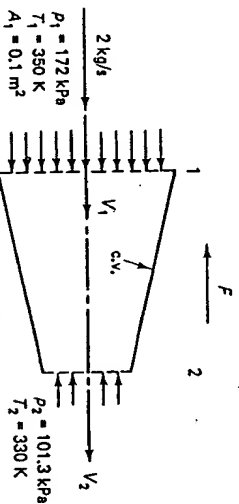


Figure 3.20

### Solution

$$\rho_1 = \frac{p_1}{RT_1} = \frac{1.72 \times 10^5}{287 \times 350} = 1.712 \text{ kg/m}^3$$

$$V_1 = \frac{2}{0.1 \times 1.712} = 11.68 \text{ m/s}$$

$$\rho_2 = \frac{p_2}{RT_2} = \frac{1.013 \times 10^5}{287 \times 330} = 1.0696 \text{ kg/m}^3$$

The energy equation is:

$$h_1 - h_2 = \frac{V_2^2}{2} - \frac{V_1^2}{2}$$

$$1000(350 - 330) = \frac{V_2^2}{2} - \frac{(11.68)^2}{2}$$

from which:

$$V_2 = 64.315 \text{ m/s}$$

The area at the exit:

$$A_2 = \frac{\dot{m}}{\rho_2 V_2} = \frac{2}{1.0696 \times 64.315} = 2.907 \times 10^{-2} \text{ m}^2$$

The force,  $F$ , acting on the control volume by the inner walls of the nozzle is given by the momentum equation as:

$$-F + p_1 A_1 - p_2 A_2 = \dot{m}(V_2 - V_1)$$

$$-F + 1.72 \times 10^5 \times 0.1 - 1.013 \times 10^5 \times 2.907 \times 10^{-2} = 2(64.315 - 11.68)$$

from which:

$$F = 14,150 \text{ N}$$

### Example 3.7

What is the percentage increase in net thrust of the rocket motor shown in Fig. 3.21 if a divergent portion of area ratio  $A_2/A^* = 1.5$  is added to the sonic nozzle? Assume steady isentropic flow, with  $c_p = 1.2 \text{ kJ/kg K}$  and  $\gamma = 1.3$ .

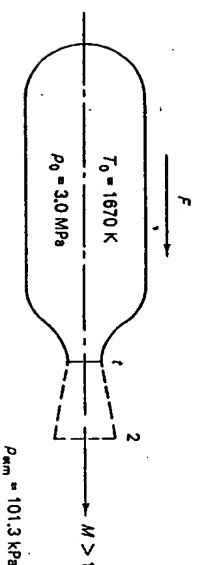


Figure 3.21

**Solution** The force  $F$  is given by:

$$F = (p_{\text{exit}} - p_{\text{atm}})A_{\text{exit}} + \dot{m}V_{\text{exit}}$$

Since the back pressure is low, the flow will be supersonic in the divergent portion of nozzle, and the properties at the throat will correspond to  $M = 1$ . Therefore:

$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2} = 1.15, \quad \text{hence } T^* = 1452 \text{ K}$$

$$\frac{p_0}{p^*} = \left(\frac{\gamma + 1}{2}\right)^{\gamma/(\gamma-1)} = (1.15)^{1.303} = 1.832, \quad \text{hence } p^* = 1.638 \text{ MPa}$$

The critical speed, which is the speed of the gas at the throat, is:

$$V^* = \sqrt{\gamma RT^*}$$

where:

$$R = \frac{\gamma - 1}{\gamma} c_p = \frac{0.3 \times 1200}{1.3} = 277 \text{ J/kg K}$$

Therefore:

$$V^* = \sqrt{(1.3) \times (277) \times (1452)} = 723 \text{ m/s}$$

Area ratio is related to Mach number by the following relation:

$$\frac{A_2}{A^*} = \frac{1}{M_2} \left[ \frac{2 + (\gamma - 1)M_2^2}{\gamma + 1} \right]^{\gamma/(\gamma-1)}$$

At an area ratio of 1.5, the Mach number can be calculated:

$$1.5 = \frac{1}{M_2} \left[ \frac{2 + 0.3M_2^2}{2.3} \right]^{1.83}, \quad \text{from which } M_2 = 1.82$$

At this Mach number the pressure ratio is:

$$\frac{p_0}{p_2} = \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)^{\gamma/(\gamma-1)} = [1 + 0.15(1.82)^2]^{1.33} = 5.96$$

Therefore, the pressure at the exit is:

$$p_2 = \frac{p_0}{5.96} = 0.503 \text{ MPa}$$

The temperature ratio is:

$$\frac{T_0}{T_2} = 1 + \frac{\gamma - 1}{2} M_2^2 = 1.497, \quad \text{hence } T_2 = 1116 \text{ K}$$

The velocity is:

$$V_2 = \sqrt{\gamma RT_2} M_2 = \sqrt{(1.3)(277)(1116)} (1.82) = 1154 \text{ m/s}$$

The mass rate of flow is:

$$\begin{aligned} \dot{m} &= \rho^* A^* V^* = \rho_2 A_2 V_2 \\ &= \left( \frac{0.503 \times 10^6}{277 \times 1116} \right) (1.54 \times 1154) \\ &= 2816.64 \text{ kg/s} \end{aligned}$$

The thrust ratio is:

$$\begin{aligned} \frac{F}{F^*} &= \frac{(p_2 - p_{\text{atm}}) \frac{A_2}{A^*} + \dot{m} \frac{V_2}{A^*}}{(p^* - p_{\text{atm}}) + \frac{\dot{m} V^*}{A^*}} \\ &= \frac{(5.03 - 1.013) \times 10^6 (1.5) + 2816.6 \times 1154}{(16.38 - 1.013) 10^5 + 2816.6 \times 723} = 1.078 \end{aligned}$$

### 3.8 REAL NOZZLES AND DIFFUSERS

Flow of fluids through nozzles and diffusers is always accompanied by frictional effects. These losses occur mainly in the boundary layer, causing irreversible effects, so that there is an increase in entropy and a corresponding decrease in stagnation pressure. According to the first and second laws of thermodynamics, the change between two stagnation states can be expressed as:

$$T_0 ds_0 = dh_0 - v_0 dp_0$$

If no heat is transferred, and if no work is done,  $dh_0 = 0$  so that,

$$T_0 ds = -v_0 dp_0 \quad (\text{where } ds = ds_0)$$

Since  $v_0 = RT_0/p_0$  for a perfect gas, this equation reduces to:

$$\frac{ds}{R} = - \frac{dp_0}{p_0}$$

When integrated, this equation becomes:

$$\frac{s_2 - s_1}{R} = - \ln \frac{p_{02}}{p_{01}} \quad (3.43)$$

According to the second law of thermodynamics, entropy always increases during this process, and there must be a corresponding reduction in stagnation pressure. To obtain a certain mass flow rate with a real nozzle, the flow area must be larger than that of the isentropic (ideal) nozzle.

Isentropic flow provides a basis for evaluating the performance of a real nozzle or a diffuser. In the case of a nozzle, the kinetic energy at the nozzle exit

compared to the kinetic energy in isentropic expansion to the same exit pressure is called *nozzle efficiency*. Let subscript 2 designate the real state of the fluid at the nozzle exit. Also, let subscript 2s designate the exit state if the fluid could expand isentropically to that exit pressure. Referring to Fig. 3.22, the nozzle efficiency is:

$$\eta_{\text{nozzle}} = \frac{\left(\frac{V_2^2}{2}\right)_{\text{real}}}{\left(\frac{V_{2s}^2}{2}\right)_{\text{isen}}} \quad (3.44)$$

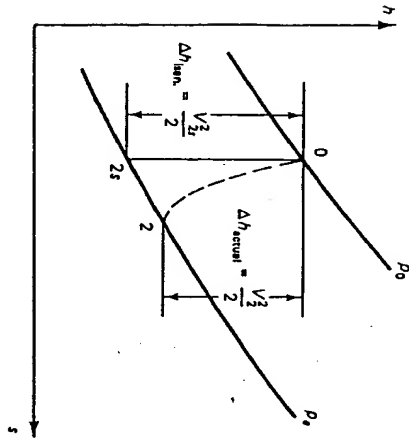


Figure 3.22 The  $h$ - $s$  diagram for flow through a nozzle.

The nozzle efficiency may be expressed, according to the first law, in terms of enthalpy:

$$\eta_{\text{nozzle}} = \frac{h_0 - h_2}{h_0 - h_{2s}} \quad (3.45)$$

where subscript 0 refers to stagnation conditions. Nozzle efficiencies generally fall in the range of 90 to 99 percent.

Similarly, the real velocity at the nozzle exit divided by that in an isentropic expansion to the same exit pressure defines the *velocity coefficient*:

$$C_v = \frac{(V_2)_{\text{real}}}{(V_{2s})_{\text{isen}}} \quad (3.46)$$

It is obvious that the velocity coefficient is related to the nozzle efficiency,

and the following equation can be derived:

$$C_v = \sqrt{\eta_{\text{nozzle}}} \quad (3.47)$$

The *coefficient of discharge* is defined as the ratio of the real mass rate of flow to the mass rate if the flow were isentropic:

$$C_d = \frac{(\dot{m})_{\text{real}}}{(\dot{m})_{\text{isen}}} \quad (3.48)$$

In diffusers, static pressure is an important parameter. Referring to Fig. 3.23, diffuser efficiency is defined as the isentropic enthalpy change if the flow is de-

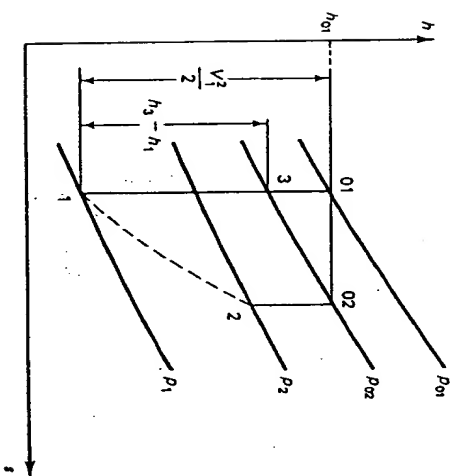


Figure 3.23 The  $h$ - $s$  diagram for flow through a diffuser.

celerated to a pressure equal to the stagnation pressure at the diffuser exit divided by the decrease in kinetic energy if the flow entering the diffuser is decelerated isentropically to the isentropic stagnation state:

$$\eta_D = \frac{h_3 - h_1}{\frac{V_1^2}{2}} = \frac{h_3 - h_1}{h_{01} - h_1} \quad (3.49)$$

Diffuser efficiency can also be expressed in terms of the Mach number of the diffuser inlet. The following relations apply to a perfect gas:

$$h = c_p T, \quad \frac{T_3}{T_1} = \left(\frac{p_{02}}{p_1}\right)^{(\gamma-1)/\gamma}, \text{ and}$$

$$c_p = \frac{\gamma R}{\gamma - 1}, \quad V_1^2 = c_1^2 M_1^2 = \gamma R T_1 M_1^2$$

Therefore, Eq. (3.49) can be expressed as:

$$\eta_D = \frac{c_p(T_3 - T_1)}{\frac{V_1^2}{2}} = \frac{c_p T_1 \left( \frac{T_3}{T_1} - 1 \right)}{\frac{V_1^2}{2}} = \frac{\left( \frac{p_{02}}{p_1} \right)^{(\gamma-1)/\gamma} - 1}{\frac{\gamma-1}{2} M_1^2} \quad (3.50)$$

But since  $(p_{02}/p_1) = (p_{02}/p_{01})(p_{01}/p_1)$  and

$$\frac{p_{01}}{p_1} = \left( 1 + \frac{\gamma-1}{2} M_1^2 \right)^{\gamma/(\gamma-1)}$$

therefore the diffuser efficiency can be expressed in terms of stagnation pressures:

$$\eta_D = \frac{\left( 1 + \frac{\gamma-1}{2} M_1^2 \right) \left( \frac{p_{02}}{p_{01}} \right)^{\gamma/(\gamma-1)} - 1}{\frac{\gamma-1}{2} M_1^2} \quad (3.51)$$

Another method of evaluating a diffuser is based on the pressure coefficient. This is the change in pressure when the flow is decelerated at the diffuser exit to the stagnation pressure  $p_{02}$ , divided by the change in pressure when the flow is decelerated isentropically to the isentropic stagnation state:

$$\text{pressure coeff.} = \frac{p_{02} - p_1}{p_{01} - p_1} \quad (3.52)$$

### Example 3.8

Air at a velocity of 210 m/s decelerates through a diffuser to a velocity of 60 m/s. The temperature and pressure at the inlet are 278 K and 80 kPa and the exit pressure is 90 kPa. Assuming one-dimensional steady flow, calculate:

- The change in stagnation pressure.
- The change in entropy.
- The diffuser efficiency.

### Solution

(a) Referring to Fig. 3.23, the energy equation is:

$$c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2}$$

from which:

$$\begin{aligned} T_2 &= T_1 + \frac{V_1^2}{2c_p} - \frac{V_2^2}{2c_p} \\ &= 278 + \frac{1}{2 \times 1000} [(210)^2 - (60)^2] \\ &= 298.25 \text{ K} \end{aligned}$$

The Mach number at the inlet is:

$$M_1 = \frac{V_1}{20.1\sqrt{T_1}} = \frac{210}{20.1\sqrt{278}} = 0.626$$

At this Mach number the pressure ratio is:

$$\frac{p_1}{p_{01}} = 0.77$$

Therefore:

$$p_{01} = \frac{80}{0.77} = 103.9 \text{ kPa}$$

The Mach number at the exit is:

$$M_2 = \frac{V_2}{20.1\sqrt{T_2}} = \frac{60}{20.1\sqrt{298.25}} = 0.173$$

The corresponding pressure ratio is:

$$\frac{p_2}{p_{02}} = 0.98$$

from which:

$$p_{02} = \frac{90}{0.98} = 91.8 \text{ kPa}$$

Therefore, the change in the stagnation pressure is:

$$p_{02} - p_{01} = 91.8 - 103.9 = -12.1 \text{ kPa}$$

(b) The change in entropy is:

$$\Delta s = R \ln \frac{p_{01}}{p_{02}} = 287 \ln \frac{103.9}{91.8} = 35.22 \text{ J/kg K}$$

(c) The diffuser efficiency is:

$$\begin{aligned} \eta_D &= \frac{\left( \frac{p_{02}}{p_1} \right)^{(\gamma-1)/\gamma} - 1}{\frac{\gamma-1}{2} M_1^2} \\ &= \frac{\left( \frac{91.8}{80} \right)^{0.41/1.4} - 1}{0.2(0.626)^2} = 0.51 \end{aligned}$$

## PROBLEMS

- 3.1. Air flows at the rate of 1 kg/s through a convergent-divergent nozzle. The entrance area is  $2 \times 10^{-3} \text{ m}^2$  and the inlet temperature and pressure are 438 K and 580 kPa. If the exit pressure is 140 kPa and the expansion is isentropic, find:
- The velocity at entrance.
  - The stagnation temperature and stagnation pressure.
  - The throat and exit areas.
  - The exit velocity.
- 3.2. A convergent nozzle has an exit area  $6.5 \times 10^{-4} \text{ m}^2$ . Air enters the nozzle at  $p_0 = 680 \text{ kPa}$ ,  $T_0 = 370 \text{ K}$ . If the flow is isentropic, determine the mass rate of flow for back pressure of:
- 359 kPa.
  - 540 kPa.
  - 200 kPa.
- 3.3. A convergent-divergent steam nozzle has an exit area of  $3.2 \times 10^{-4} \text{ m}^2$  and an exit pressure of 270 kPa. The inlet conditions are 1 MPa and 590 K with negligible velocity. Assume ideal flow (i.e., no losses) and

$$\frac{p^*}{p_0} = 0.545$$

Find:

- The mass rate of flow for this nozzle.
  - The throat area.
  - The sonic velocity at the throat.
- 3.4. Air flows isentropically through a convergent-divergent passage with inlet area  $5.2 \text{ cm}^2$ , minimum area  $3.2 \text{ cm}^2$  and exit area  $3.87 \text{ cm}^2$ . At the inlet the air velocity is 100 m/s, pressure is 680 kPa, and temperature 345 K. Determine:
- The mass rate of flow through the nozzle.
  - The Mach number at the minimum-area section.
  - The velocity and the pressure at the exit section.
- 3.5. Air is flowing in a convergent nozzle. At a particular location within the nozzle the pressure is 280 kPa, the stream temperature is 345 K, and the velocity is 150 m/s. If the cross-sectional area at this location is  $9.29 \times 10^{-3} \text{ m}^2$ , find:
- The Mach number at this location.
  - The stagnation temperature and pressure.
  - The area, pressure, and temperature at the exit where  $M = 1.0$ .
  - The mass rate of flow for the nozzle.
- Indicate any assumptions you may make and the source of data used in the solution.
- 3.6. Air flows isentropically at the rate of 0.5 kg/s through a supersonic convergent-divergent nozzle. At the inlet, the pressure is 680 kPa, the temperature 295 K, and the area is  $6.5 \text{ cm}^2$ . If the exit area is  $13 \text{ cm}^2$ , calculate:
- The stagnation pressure and temperature.
  - The exit Mach number.
  - The exit pressure and temperature.
  - The area and the velocity at the throat.

- What will be the maximum rate of flow and the corresponding exit Mach number if the flow is completely subsonic in the nozzle?
- 3.7. A stream of carbon dioxide is flowing in a 7.5 cm I.D. pipe at a stream pressure of 680 kPa and a stream temperature of 365 K. A 7.5 cm  $\times$  5 cm venturimeter installed in this pipe shows a pressure differential reading of 168 mm Hg. Assuming ideal flow, determine:
- The mass rate of flow of  $\text{CO}_2$ . Compare your answer with that obtained if the gas is considered incompressible.
  - If the mass rate of flow of  $\text{CO}_2$  were to be doubled, what would be the new pressure differential reading for the venturimeter?
  - If the fluid were hydrogen instead of  $\text{CO}_2$ , other conditions being the same as given in the problem statement, what would be the mass rate of flow?
  - If the temperature of the  $\text{CO}_2$  were 440 K instead of 365 K, other conditions being the same as given in the problem statement, what would be the mass rate of flow for the  $\text{CO}_2$ ?
- 3.8. A 0.14 m<sup>3</sup> tank of compressed air discharges through a 2.2 cm diameter converging nozzle located in the side of the tank. If the mass flow coefficient of the nozzle based on isentropic flow through it is 0.95 and the gas within the tank expands isothermally from 1 MPa to 350 kPa, plot the pressure in the tank versus elapsed time as the pressure decreases. Assume the temperature of the tank is 295 K and the surrounding pressure is 101.3 kPa.
- 3.9. Air at stagnation conditions of 2 MPa and 750 K flows isentropically through a converging-divergent nozzle. If the maximum flow rate is 5.4 kg/s, determine:
- The throat area in m<sup>2</sup>.
  - The velocity, pressure, and temperature at the nozzle exit if the exit area is three times as large as the throat area.
- 3.10. Find the throat and exit areas in m<sup>2</sup> for a critical-flow nozzle handling air at the rate of 6.7 kg/s when the desired exit velocity is 1100 m/s with the stream at  $p = 170 \text{ kPa}$  and  $T = 310 \text{ K}$ . Assume isentropic flow and  $\gamma = 1.4$ .
- 3.11. Air flows reversibly and adiabatically in a nozzle. At section 1 of the nozzle, the velocity, pressure, temperature, and area are 165 m/s, 350 kPa, 480 K, and  $13 \times 10^{-4} \text{ m}^2$ . At section 2 in nozzle the area is  $26 \times 10^{-4} \text{ m}^2$ . Find:
- The mass flow rate in the nozzle.
  - $V_2$ ,  $M_2$ ,  $p_2$ ,  $T_2$  and  $v_2$ .
- (Note: There are two independent answers for this condition. Calculate both cases. If there is a throat, determine its area.)
- 3.12. Air at a pressure of 680 kPa and a temperature of 833 K enters a converging-diverging nozzle through a line of  $4.6 \times 10^{-3} \text{ m}^2$  area and expands to a delivery-region pressure of 33 kPa. Assuming isentropic expansion and a mass rate of flow of 1 kg/s, find:
- The stagnation enthalpy.
  - The temperature and enthalpy at discharge.
  - The Mach number and velocity of the air stream at discharge.
  - The maximum mass flow rate per unit area.
- 3.13. Air flows isentropically at the rate of 1 kg/s through a duct. At one section of the duct the cross-sectional area is  $9.3 \times 10^{-3} \text{ m}^2$ , static pressure is 200 kPa, and stagnation temperature is 550 K. Determine the velocity of the stream and the minimum area at the exit of the duct that causes no reduction in the mass rate of flow.

- 3.14. Air flows isentropically through a converging nozzle. At the inlet of the nozzle the pressure  $p_1 = 340$  kPa, the temperature  $T_1$  is 550 K, the velocity  $V_1$  is 200 m/s, and the cross-sectional area  $A_1$  is  $9.3 \times 10^{-3}$  m<sup>2</sup>. Consider air to be an ideal gas with  $\gamma = 1.4$  and find:

- The stagnation temperature and pressure.
- The sonic velocity and the Mach number at the inlet.
- The area, pressure, temperature, and velocity at the exit if  $M = 1$  at exit.
- Draw graphs of  $G$ ,  $M$ ,  $V$ , and  $v$  versus pressure, indicating the values at the inlet and exit of the nozzle.

- 3.15. Superheated steam expands isentropically in a convergent-divergent nozzle from an initial state in which the pressure is 2.0 MPa and the superheat is 378 K to a pressure of 680 kPa. The rate of flow is 0.5 kg/s.

- Find the velocity of the steam and the cross-sectional area of the nozzle at the sections where the pressures are 1.0 MPa and 1.2 MPa.
- Determine the pressure, velocity, and cross-sectional area at the throat.
- Determine the velocity and cross-sectional area at discharge.

Assume that  $p^*/p_0 = 0.55$ .

- 3.16. A convergent nozzle receives steam at a pressure of 3.4 MPa and a temperature of 640 K with negligible velocity. The nozzle discharges into a chamber at which the pressure is maintained at 1.36 MPa. If the throat area of the nozzle is  $2.3 \times 10^{-4}$  m<sup>2</sup> and the discharge chamber area is 0.056 m<sup>2</sup>, find:

- The velocity at the throat.
- The mass rate of flow.

Assume that  $p^*/p_0 = 0.55$  and the flow is isentropic.

- 3.17. Air flows isentropically through the convergent-divergent nozzle shown in Fig. 3.24. The inlet pressure is 80 kPa, the inlet temperature 295 K, and the back pressure 1.013 kPa. What should be the exit diameter of the nozzle which corresponds to the maximum obtainable value of Mach number at the exit? What are the mass rate of flow, the exit Mach number, and the exit temperature?

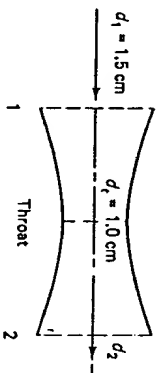


Figure 3.24

- 3.18. Air flows isentropically through a convergent-divergent nozzle at the rate of 30 kg/s. The stagnation temperature is 230°C and the stagnation pressure is 200 kPa. If the Mach number at the exit is 2.5, calculate the nozzle throat area and area, pressure, temperature, and velocity at the exit.

- 3.19. Air flows isentropically in a convergent-divergent nozzle with supersonic flow at the exit. The inlet and exit areas are each equal to 8 cm<sup>2</sup> and the throat area is 4 cm<sup>2</sup>. If the pressure at the throat is 200 kPa, determine the inlet and exit pressures.

- 3.20. A rocket motor is fitted with a convergent-divergent nozzle having a throat diameter 2.5 cm. If the chamber pressure is 1 MPa and the chamber temperature is 2200 K, determine:

- The mass flow rate through the nozzle.
- The Mach number at the exit ( $p_{\text{back}} = 101.3$  kPa).
- The thrust developed at sea level.

Assume that the products of combustion behave like a perfect gas ( $\gamma = 1.4$ ,  $R = 540$  J/kg K), and the expansion through the nozzle is isentropic.

- 3.21. Air is flowing through a section of a straight convergent nozzle. At the entrance to the nozzle section the area is  $4 \times 10^{-3}$  m<sup>2</sup>, the velocity is 100 m/s, the air pressure is 680 kPa, and the air temperature is 365 K. At the exit of the section the area is  $2 \times 10^{-3}$  m<sup>2</sup>. Assume reversible adiabatic flow. Calculate the magnitude and direction of the force exerted by the fluid upon the given nozzle section.

- 3.22. In order to provide thrust-vector control for a space vehicle, nitrogen at a stagnation pressure of 2.7 MPa and a stagnation temperature 295 K expands isentropically through a nozzle. If the back pressure is 70 kPa and the flow rate is 0.05 kg/s, determine:

- The maximum thrust developed.
- The throat area of the nozzle.
- The exit area of the nozzle.

- 3.23. A rocket motor is being tested at sea level where the pressure is 100 kPa. The chamber pressure  $p_0 = 1.2$  MPa, the chamber temperature  $T_0 = 3000$  K, and the throat of the nozzle has an area of 8 cm<sup>2</sup>. If the ratio of specific heats  $\gamma = 1.25$  and the gas constant  $R = 380$  J/kg K, determine:

- The exit area and exit velocity for isentropic expansion in the nozzle.
- The thrust developed.
- If the exit area is reduced by 10 percent, resulting in an underexpanded nozzle, what will then be the thrust?

- 3.24. Solve Example 3.6 if the flow were isentropic with the same inlet conditions and exit pressure. What is the temperature at the exit?

- 3.25. Air at a stagnation temperature  $T_0 = 350$  K flows isentropically through a diffuser. The diffuser diameter at the inlet is 6 cm and at the exit it is 12 cm. The pressure and temperature at the inlet are 70 kPa and 330 K. Determine the force that the fluid exerts on the diffuser walls.

- 3.26. The supersonic nozzle of a rocket motor is designed to operate at sea level ( $p_{\text{amb}} = 101.3$  kPa). If the chamber pressure and temperature are 2 MPa and 1500 K, determine the ratio of the thrust at sea level to the thrust in outer space ( $p_{\text{amb}} = 0$  kPa). Assume the rocket exhaust gases to behave as a perfect gas with  $\gamma = 1.4$  and  $R = 0.3$  kJ/kg K.

- 3.27. Nitrogen at a stagnation pressure of 2.5 MPa and a stagnation temperature of 1400 K expands in a nozzle. If the ambient pressure is 40 kPa, what is the exit Mach number for optimum thrust? If the flow rate is 5 kg/s, what is the throat area? exit area?